# Homotopy theory

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# Part I

# Characteristic Classes

# Chapter 1

# Stiefel-Whitney Classes

## 1.1 Axiomatic definitions

**Definition 1.1.1.** For a topological space $B$ and a $n$-dimensional vector bundle $B(\xi) = p : E \to B$ ($B(\xi)$ will denote the base space of the bundle later when we want to emphasize the base space of the bundle), we define the *Stiefel-Whitney classes* $w_i(\xi)$ of the bundle to be a collection of modulo 2 cohomology classes having the following properties:

1. $w_i(\xi) \in H^i(B(\xi); \mathbb{Z}/2\mathbb{Z})$, in particular $w_0(\xi) = 1 \in H^0(B(\xi); \mathbb{Z}/2\mathbb{Z})$, and $w_i(\xi) = 0$ for $i > n$.

2. The Stiefel-Whitney classes are natural. That is, for a bundle map $f : B(\xi) \to B(\eta)$, we have $w_i(\xi) = f^* w_i(\eta)$.

3. The Stiefel-Whitney classes of the direct sum bundle can be computed as follows:

$$w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_i(\xi) \smile w_{k-i}(\eta)$$

4. For the canonical line bundle over $\mathbb{RP}^1$, $\gamma_1^1$, we have $w_1(\gamma_1^1)$ nontrivial.

*Remark* 1. Even though these definitions works well for general topological spaces, we shall temporarily use them only on manifolds, where the notion of vector bundles works best. As the most natural bundle of a manifold $M$, the tangent bundle $\tau_M$, the Stiefel-Whitney class of $\tau_M$ is often denoted by $w(M)$.

**Definition 1.1.2.** We define $H^{\Pi}(X; \mathbb{Z}/2\mathbb{Z})$ to be the ring consisting of all formal infinite series

$$a = a_0 + a_1 + \dots$$

where $a_i \in H^i(X; \mathbb{Z}/2\mathbb{Z})$, and the obvious product defined on it. Then we define the total Stiefel-Whitney class to be $w(\xi) = w_0(\xi) + w_1(\xi) + \dots$.
Under this definition, the third axiom could be written simply as $w(\xi \oplus \eta) = w(\xi)w(\eta)$.

*Remark* 2. This ring has properties similar to the formal series ring. For example the units of this ring are $a = 1 + a_1 + \dots$.

We have the following immediate consequences of the axioms:

**Corollary 1.1.3.** *1. If $\xi \simeq \eta$, then $w_i(\xi) = w_i(\eta)$ under the natural equivalence of the cohomology groups.*

*2. If $\varepsilon$ is trivial, then $w_i(\varepsilon) = 0$ for $i > 0$.*

3. *If $\varepsilon$ is trivial, then for any vector bundle $\xi$, $w_i(\xi \oplus \varepsilon) = w_i(\xi)$.*

4. *If $\xi$ is an $n$-dimensional bundle equipped with an Euclidean metric having a non-vanishing section, then $w_n(\xi) = 0$.*

5. *$w(\xi)$ are units in $H^{\Pi}(B(\xi); \mathbb{Z}/2\mathbb{Z})$.*

*Proof.* Only the fourth property is non-trivial. To see this, let $\varepsilon$ be the trivial bundle generated by the non-vanishing section, then notice $\xi$ splits as $\varepsilon \oplus \varepsilon^{\perp}$. □

**Definition 1.1.4.** Being a unit, the inverse of $w(\xi)$ in $H^{\Pi}(B(\xi); \mathbb{Z}/2\mathbb{Z})$ is defined to be $\bar{w}(\xi)$.

**Corollary 1.1.5.** *If we have $\xi \oplus \eta$ a trivial bundle on $B$, then $w(\xi) = \bar{w}(\eta)$.*

*Remark* 3. Often, the above corollary is used when $\xi$ is the tangent bundle and $\eta$ the normal bundle.

## 1.2 Elementary computations in $S^n$ and $\mathbb{RP}^n$

**Corollary 1.2.1.** $w(\tau_{S^n}) = 1$.

*Proof.* Consider the embedding $S^n \hookrightarrow \mathbb{R}^{n+1}$. Then we have the normal bundle $\nu$ and the direct sum $\nu \oplus \tau$ both trivial. □

**Proposition 1.2.2.** *We have $H^*(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[a]$, where the degree of $x$ is one.*

*Proof.* We use the standard cell structure on $\mathbb{RP}^n$. Then, since the mapping degree of the characteristic map of the cells are 2, we have the map $C^i \to C^{i+1}$ the trivial map. This implies $H^i(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ for $0 \leq i \leq n$. The cup product structure follows directly. □

**Proposition 1.2.3.** *The Stiefel-Whitney class of the canonical line bundle $\gamma_n^1$ of $\mathbb{RP}^n$ is given by $w(\gamma_n^1) = 1 + a$.*

*Proof.* We consider the standard inclusion $j : \mathbb{RP}^1 \to \mathbb{RP}^n$, covered by the bundle inclusion $\gamma_1^1 \to \gamma_n^1$. Thus $j^* w_1(\gamma_n^1) = w_1(\gamma_1^1) \neq 0$, and since the bundle is 1-dimensional, the higher Stiefel-Whitney classes vanish. □

**Corollary 1.2.4.** *Recall that the trivial line bundle of $\mathbb{RP}^n$ embeds into the $n+1$-dimensional trivial bundle $\gamma_n^1 \hookrightarrow \varepsilon_{n+1}$. Thus we have the Stiefel-Whitney classes of the orthogonal complement of $w(\gamma^{\perp}) = 1 + a + a^2 + \cdots + a^n$.*

We finish this section with the computations of the Stiefel-Whitney classes of the tangent bundle of $\mathbb{RP}^n$.

**Lemma 1.2.5.** *We have the isomorphism $\tau_{\mathbb{RP}^n} \simeq \mathrm{Hom}(\gamma_n^1, \gamma^{\perp})$.*

*Proof.* A direct check using the standard construction. □

**Lemma 1.2.6.** *We have $\tau \oplus \varepsilon^1 \simeq (\gamma_n^1)^{\oplus(n+1)}$.*

*Proof.* By the preceding lemma, we have

$$\begin{aligned}
&\tau \oplus \varepsilon^1 \\
\simeq &\mathrm{Hom}(\gamma_n^1, \gamma^{\perp}) \oplus \mathrm{Hom}(\gamma_n^1, \gamma_n^1) \\
\simeq &\mathrm{Hom}(\gamma_n^1, \gamma^{\perp} \oplus \gamma_n^1) \\
\simeq &\mathrm{Hom}(\gamma_n^1, \varepsilon^{n+1}) \\
\simeq &\mathrm{Hom}(\gamma_n^1, \varepsilon)^{\oplus(n+1)} \\
\simeq &(\gamma_n^1)^{\oplus(n+1)}
\end{aligned}$$

□

**Corollary 1.2.7.**

$$w(\mathbb{RP}^n) = (1+a)^{n+1}$$

*Proof.* Directly compute using the preceding lemma. $\square$

## 1.3 The Thom isomorphism

**Definition 1.3.1.** For a vector bundle $\xi = \pi : E \to B$, we denote $E_0$ the subspace of $E$ consisting of all the non-zero elements within each fibre. If we let $F$ be the fibre, then $F_0$ is similarly defined as $F - \{0\}$.
We say this bundle is $R$-oriented if there is a family of $R$-coefficient cohomology class $u_x \in H^n(F_x, F_0; R)$ which is a generator and agrees with each other in the locally trivializable condition.
If there is a class $u \in H^n(E, E_0; R)$ such that $u|_x = u_x$, we call $u$ the fundamental class of the bundle.
Equivalently, we may consider the unit ball bundle and unit sphere bundle associated to $\xi$, say $B(\xi)$ and $S(\xi)$. Then since $(E, E_0)$ is homotopy equivalent to $(B(\xi), S(\xi))$, we may equivalently define the fundamental class in $H^n(B(\xi), S(\xi); R)$.

The main purpose of this section is to prove the Thom isomorphism theorem, and use it to derive some important properties:

**Theorem 1.3.2.** *For an oriented $R$-bundle $\xi = p : E \to B$ having fundamental class $u \in H^n(E, E_0, R)$, we have an isomorphism $\smile u : H^i(E; R) \to H^{n+i}(E, E_0; R)$. We define $\varphi : H^i(B; R) \to H^{n+i}(E, E_0; R)$ be the canonical isomorphism composed with the isomorphism $p^* : H^i(B; R) \to H^i(E; R)$. In particular, since $\mathbb{Z}/2\mathbb{Z}$-bundles are always oriented, the above isomorphism is always available. We may also substitute all the above $H^*(E, E_0; R)$ into $H^*(B(\xi), S(\xi); R)$.*

*Remark* 4. For a vector bundle $\xi = p : E \to B$, we may define its Thom space $\mathrm{Th}(\xi)$ to be the one-point compactification of $E$. Then, notice that $\widetilde{H}^*(\mathrm{Th}(\xi)) \simeq H^*(E, E_0)$. Therefore the Thom isomorphism can also be expressed by $H^*(B) \simeq H^{*+n}(\mathrm{Th}(\xi))$.

We first recall the Leray-Hirsch theorem.

**Theorem 1.3.3.** *Suppose we have a fibration $F \hookrightarrow E \twoheadrightarrow B$ and its subfibration $F' \hookrightarrow E' \twoheadrightarrow B$, such that $H^i(F, F'; R)$ is a finitely generated free $R$-module for each $i$. Assume further that there is are classes $c_j \in H^*(E, E'; R)$ whose restrictions form a basis for $H^*(F, F'; R)$, then $H^*(E, E'; R)$ is a free $H^*(B; R)$ module with basis $\{c_j\}$, where $bc := p^*(b) \smile c$.*

We use this to prove the second part of the Thom isomorphism theorem directly.

*Proof.* Notice $H^i(F, F'; R) = R$ when $i = 0, n$ and is 0 otherwise. Thus $u \in H^*(E, E'; R)$ is a basis for $H^*(F, F'; R)$, and thus is a $H^*(B; R)$ basis of $H^*(E, E'; R)$. $\square$

Then, we prove following property:

**Proposition 1.3.4.** *If $\xi$ is an $R$-oriented bundle, suppose either $R$ is a field or $B$ is compact, then the fundamental class always exists.*

*Proof.* We prove there is a unique $u \in H^n(E, E_0; R)$ that restricts to $u_x$. We finish the proof in four steps.
The first step: when $\xi$ is a trivial bundle, this is obvious.
The second step: when $B = B_1 \cup B_2$, and $\xi$ is a trivial bundle when restricted to both $B_1$ and

$B_2$, we let $E^1 = p^{-1}(B_1)$ and $E^2 = p^{-1}(B_2)$. Then, we have the exact sequence obtained by the Mayer-Vietoris sequence below:

$$H^n(E, E_0; R) \longrightarrow H^n(E^1, E_0^1; R) \oplus H^n(E^2, E_0^2; R) \longrightarrow H^n(E^1 \cap E^2, E_0^1 \cap E_0^2; R)$$

Now that we have a class $u_1$ and $u_2$ in $H^n(E^1, E_0^1; R)$ and $H^n(E^2, E_0^2; R)$ respectively, their restriction to $H^n(E^1 \cap E^2, E_0^1 \cap E_0^2; R)$ coincides by the uniqueness condition on $\xi|_{B_1 \cap B_2}$ (note this is a trivial bundle). Thus by exactness, there is a class $u \in H^n(E, E_0; R)$ that restricts to $u_1$ and $u_2$, and thus restricts to all $u_x$.
The third step: when $B$ is compact, we cover $B$ by open subsets $B_i$ such that $\xi$ is trivializable over $B_i$. Then by compactness, we can find a finite subcover of $B_i$. Then by the second step, we have the uniqueness and existence.
Note here that we have already proven the proposition on compact spaces. The fourth step: when $B$ is arbitrary, we attempt to prove $H^n(B; R) = \varprojlim_{C \subseteq B} H^n(C; R)$, where $C$ is compact in $B$ and ordered by inclusion. To prove this, first notice that we have $\varinjlim H_n(C; R) \to H_n(B; R)$ is an isomorphism since homology is naturally compactly supported. Then, for field coefficients, $H^n(X; R)$ is naturally isomorphic to $\mathrm{Hom}(H_n(X; R); R)$ by the universal coefficient theorem applied to $R$-coefficient homology and cohomology, where the $\mathrm{Ext}(H_{n-1}(X; R), R)$ term vanishes since $H_{n-1}(X; R)$ is a free $R$-module. Thus

$$\begin{aligned}
&H^n(B; R) \\
=&\mathrm{Hom}(H_n(B; R), R) \\
\simeq&\mathrm{Hom}(\varinjlim H_n(C; R), R) \\
\simeq&\varprojlim \mathrm{Hom}(H_n(C; R), R) \\
=&\varprojlim H^n(C; R)
\end{aligned}$$

Similarly $H^n(E, E_0; R) = \varprojlim H^n(p^{-1}(C), p^{-1}(C)_0; R)$ since obviously every compact subspace of $(E, E_0)$ is contained in some $(p^{-1}(C), p^{-1}(C)_0)$. Then the classes in $H^n(p^{-1}(C), p^{-1}(C)_0; R)$ obviously agrees with inclusions by the uniqueness in compact case, thus put together to a class $u \in H^n(E, E_0; R)$. □

With another technical lemma, we prove the case for arbitrary coefficients and base spaces.

**Lemma 1.3.5.** *Let $K$ and $K'$ be free complexes over $\mathbb{Z}$, and $f : K \to K'$ a chain map of degree $d$. Then if $f : H^*(K', F) \to H^*(K, F)$ is an isomorphism for every field $K$, then $f$ induces isomorphisms of homology and cohomology with arbitrary coefficients.*

*Proof.* Without loss of generality, we suppose $d = 0$. Recall the construction of the mapping cone $K^f$. It then fits in an exact sequence of complexes $0 \longrightarrow K' \longrightarrow K^f \longrightarrow K \longrightarrow 0$. We prove that $H_n(K^f) = 0$ for all $n$. Firstly, since $f : H^*(K'; F) \to H^*(K; F)$ is an isomorphism, we have $f^* : \mathrm{Hom}(H_*(K' \otimes F), F) \to \mathrm{Hom}(H_*(K \otimes F), F)$ an isomorphism, thus $\mathrm{Hom}(H_*(K^f \otimes F), F) = 0$ by the long exact sequence. Therefore $H_n(K^f \otimes F) = 0$. In particular, $H_n(K^f \otimes \mathbb{Q}) = 0$, and then it suffices to prove the torsion group $H_n(K^f) = 0$. This follows from considering $H_n(K^f \otimes \mathbb{F}_p) = 0$ to eliminate $p$-torsion parts. □

**Proposition 1.3.6.** *For a bundle $\xi$, if $u$ is the fundamental class of $\xi$, then the correspondence $u \frown : H_{n+i}(E, E_0; \mathbb{Z}) \to H_i(E; \mathbb{Z})$ is an isomorphism.*

*Proof.* We pick a cochain class $v$ representing $u$. Note $v \frown : C_{n+i}(E, E_0; \mathbb{Z}) \to C_i(E; \mathbb{Z})$ is a chain map. Then, using the identity $\langle c, v \frown d \rangle = \langle c \smile v, d \rangle$ (here $c$ is a cochain class and $d$ a chain class), we have the cohomology mapping induced by $v \frown$ is $\smile u$. Then the conclusion follows from the preceding lemma. □

We are finally ready to prove the full version of proposition 1.3.4

**Theorem 1.3.7.** *If $\xi$ is an $R$-oriented bundle, then the fundamental class always exists.*

*Proof.* The first three steps are the same as the previous partial proof. For the fourth step, we have $H_{n-1}(E, E_0; \mathbb{Z}) \simeq H_{-1}(E; \mathbb{Z}) = 0$, and thus $H^n(E, E_0; R) = \mathrm{Hom}(H_n(E, E_0; \mathbb{Z}), R)$. The rest is also the same. □

**Corollary 1.3.8.** *We have a fundamental class in $u \in H^n(E, E_0; \mathbb{Z}/2\mathbb{Z})$ inducing the Thom isomorphism $\smile u : H^i(E; \mathbb{Z}/2\mathbb{Z}) \to H^i(E, E_0; \mathbb{Z}/2\mathbb{Z})$.*

*Proof.* A bundle is always $\mathbb{Z}/2\mathbb{Z}$ oriented. □

## 1.4 Existence and uniqueness of the Stiefel-Whitney classes

Before we construct the Stiefel-Whitney classes explicitly, we first review some properties of the Steenrod operations of cohomology.

**Theorem 1.4.1.** *There exists homomorphisms $Sq^i : H^n(X, Y; \mathbb{Z}/2\mathbb{Z}) \to H^{n+i}(X, Y; \mathbb{Z}/2\mathbb{Z})$ which can be defined using the following axioms:*

1. $Sq^0 = \mathrm{id}$.
2. *$Sq^i$ is natural.*
3. $Sq^i(\alpha \smile \beta) = \sum_{j=0}^{i} Sq^j(\alpha) \smile Sq^{i-j}(\beta)$.
4. *$\Sigma Sq^i = Sq^i \Sigma$, where $\Sigma$ is the natural suspension isomorphism $H^n(X, Y; \mathbb{Z}/2\mathbb{Z}) \to H^{n+1}(\Sigma X, \Sigma Y; \mathbb{Z}/2\mathbb{Z})$.*
5. *$Sq^i(\alpha) = \alpha^2$ if $i = n$ for $\alpha \in H^n(X, Y; \mathbb{Z}/2\mathbb{Z})$ and is $0$ for $i > n$.*
6. *$Sq^1$ coincides with the Bockstein homomorphism associated to the sequence*
$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$
7. *$Sq^i$ satisfies the Adem relations*
$$Sq^a Sq^b = \sum_j \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j, \ a < 2b$$

*Moreover, we define the total steenrod square*
$$Sq(a) = Sq^0(a) + \cdots + Sq^n(a)$$
*for $a \in H^n(X, Y; \mathbb{Z}/2\mathbb{Z})$.*

**Construction 1.4.2.** For a vector bundle $\xi = \pi : E \to B$, we let $w_i(\xi) = \varphi^{-1} Sq^i \varphi(1) = \varphi^{-1} Sq^i(u) \in H^i(B; \mathbb{Z}/2\mathbb{Z})$, where $\varphi : H^i(B; \mathbb{Z}/2\mathbb{Z}) \to H^{i+n}(E, E_0; \mathbb{Z}/2\mathbb{Z})$ is the Thom isomorphism (theorem 1.3.2), $Sq^i$ the Steenrod square operation, and $u$ the fundamental class. Or, equivalently, we define $w(\xi) = \varphi^{-1} Sq \varphi(1) = \varphi^{-1} Sq(u)$.
We verify the axioms.

1. Obviously $w_i(\xi) \in H^i(B; \mathbb{Z}/2\mathbb{Z})$, and $w_0 = 1$, $w_i = 0$ for $i > n$.
2. Since both the Thom isomorphism and the Steenrod squares are natural.

3. Recall that for $\xi = p : E \to B$ and $\xi' = p' : E' \to B$, we have the definition of $\xi \oplus \xi'$ the pullback bundle of $\xi \times \xi' = p \times p' : E \times E' \to B \times B$ along $\Delta : B \to B \times B$. So to verify the third axiom, we first calculate the Stiefel-Whitney class of $\xi \times \xi'$.
Consider the fundamental classes $u \in H^m(E, E_0; \mathbb{Z}/2\mathbb{Z})$ and $u' \in H^n(E', E_0'; \mathbb{Z}/2\mathbb{Z})$. Thus we have the cross product $u \times u' = \pi_1^*(u) \smile \pi_2^*(u') \in H^{m+n}(E \times E', E \times E_0' \cup E_0 \times E'; \mathbb{Z}/2\mathbb{Z})$. Notice then that $E \times E_0' \cup E_0 \times E'$ are exactly the non-zero vectors in $E \times E'$, and we claim that $u \times u'$ is the fundamental class of $\xi \times \xi'$. This is verified directly by the naturality of cross products.
Then, by the construction of the Thom isomorphism, we have $\varphi(\alpha \times \alpha') = \varphi(\alpha) \times \varphi(\alpha')$, where $\varphi$ denotes the three distinct Thom isomorphisms (note that there are no issues with signs since we are working modulo 2).
Thus finally, we have
$$w(\xi \times \xi') = \varphi^{-1}(Sq(u \times u')) = \varphi^{-1}(Sq(u)) \times \varphi^{-1}(Sq(u')) = w(\xi) \times w(\xi')$$
Pulling back along the diagonal gives $w(\xi \oplus \xi') = w(\xi) \smile w(\xi')$.

4. Let $\gamma_1^1$ be the twisted line bundle over $\mathbb{RP}^1$ as usual, then the vectors with length $\leq 1$ in $E = E(\gamma_1^1)$ is obviously a mobius band $M$. We suppose it is bound by the circle $\partial M$. Since $M$ is a deformation retract of $E$, and $\partial M$ a deformation retract of $E_0$, we have $H^*(M, \partial M; \mathbb{Z}/2\mathbb{Z}) \simeq H^*(E, E_0; \mathbb{Z}/2\mathbb{Z})$. On the other hand, by definition $H^*(\mathbb{RP}^2, \mathbb{D}^2; \mathbb{Z}/2\mathbb{Z}/2\mathbb{Z}) \simeq H^*(M, \partial M)$. Since $\mathbb{D}^2$ is contractible, we have
$H^i(E, E_0; \mathbb{Z}/2\mathbb{Z}) \simeq H^i(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z})$, and thus the fundamental class $u$ must corresponds to the only non-zero generator $a \in H^1(\mathbb{RP}^2)$. Thus $Sq^1(u) = u \smile u$ corresponds to $a \smile a$. But since $a \smile a \neq 0$, $w_1(\gamma_1^1) \neq 0$.

Now we prove the uniqueness. But before this, we need to cite an important theorem.

**Definition 1.4.3.** For positive integers $n$ and $k$ and a field $\mathbb{F}$(usually $\mathbb{F}$ is equal to $\mathbb{R}$ or $\mathbb{C}$), we define the space $Gr_n(\mathbb{F}^{n+k})$ to be the quotient space $\mathrm{Hom}_{\mathbb{F}}(\mathbb{F}^n, \mathbb{F}^{n+k})/R$, where $R$ is the equivalence relation where $f = g$ iff $\mathrm{im}\, f = \mathrm{im}\, g$.
Then we define $Gr_n(\mathbb{F}) := Gr_n(\mathbb{F}^\infty) = \varinjlim_k Gr_n(\mathbb{F}^{n+k})$. Notice that we have a $n$-bundle on $Gr_n(\mathbb{F}^{n+k})$ constructed as follows. Let $X = \{(f, x) \in Gr_n(\mathbb{F}^{n+k}) \times \mathbb{F}^{n+k} | x \in \mathrm{im}\, f\}$, then $(f, x) \mapsto f$ is an $n$-bundle on $Gr_n(\mathbb{F}^{n+k})$. This put together to be an $n$-bundle on $Gr_n(\mathbb{F})$, which we will denote by $\eta^n$.

**Theorem 1.4.4.** *For a paracompact topological space $X$, an $n$-dimensional vector bundle $p : E \to X$ corresponds to a homotopy class of maps $f : X \to Gr_n(\mathbb{R})$. Moreover, this bundle is the pullback bundle of the canonical $n$-bundle $\eta^n$ over $Gr_n(\mathbb{R})$.*
*Or equivalently, we have a homotopy equivalence $Gr_n(\mathbb{R}) \simeq BO(n)$.*

Then, we have an important observation that will allow us to finish the proof.

**Theorem 1.4.5.** *We have the cohomology $H^*(Gr_n(\mathbb{R}); \mathbb{Z}/2\mathbb{Z})$ freely generated as a polynomial over $\mathbb{Z}/2\mathbb{Z}$ by the Stiefel-Whitney classes of the canonical $n$-bundle $\eta^n$.*

*Proof.* We first prove the Stiefel-Whitney classes $w_i(\eta^n)$ are algebraically independent. We consider the classifying map of the $n$-fold product $\xi = \gamma^1 \times \cdots \times \gamma^1 = \pi_1^{-1}(\gamma^1) \oplus \cdots \oplus \pi_n^{-1}(\gamma^1)$ on the space $\mathbb{RP}^\infty \times \cdots \times \mathbb{RP}^\infty$. Now, the classifying map $g : \mathbb{RP}^\infty \times \cdots \times \mathbb{RP}^\infty \to Gr_n(\mathbb{R})$ gives $w_i(\xi) = g^* w_i(\eta^n)$. Recall $H^*(\mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[a]$ on one generator having degree 1, thus $H^*(\mathbb{RP}^\infty \times \cdots \times \mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[a_1, \ldots, a_n]$, all having degree 1. But note that $w(\xi) = w(\gamma^1)^n = (1 + a_1) \cdots (1 + a_n)$, thus $w_i(\xi)$ is the $i$-th symmetric polynomial. But it is well-known that they are algebraically independent, and thus $w_i(\eta^n)$ must be algebraically independent.

Then, we prove that they indeed generate the whole cohomology group of $H^*(Gr_n(\mathbb{R});\mathbb{Z}/2\mathbb{Z})$. This is by the canonical CW decomposition of $Gr_n(\mathbb{R})$. Since the number of $m$-cells are equal to the number of sequences $1 \le \sigma_1 \le \cdots \le \sigma_n \le m$. But the number of such sequences are exactly equal to the number of monomials formed by $w_i(\eta^n)$ of degree $m$. Thus $w_i(\eta^n)$ freely generate $H^*(Gr_n(\mathbb{R});\mathbb{Z}/2\mathbb{Z})$.
Finally, we are ready to prove the uniqueness. Firstly, since the canonical bundle $\gamma_1^1$ of $\mathbb{RP}^1$ has Stiefel-Whitney class $1+a$, the canonical bundle $\gamma^1$ of $\mathbb{RP}^\infty$ has Stiefel-Whitney class $1+a$ by the canonical inclusion. Then, let the $n$-fold product $\xi = \gamma^1 \times \cdots \times \gamma^1$, we have $w(\xi) = (1+a_1)\cdots(1+a_n)$. Now, since we have a bundle map $\xi \to \eta^n$, since $H^*(Gr_n(\mathbb{R});\mathbb{Z}/2\mathbb{Z})$ maps injectively into $H^*(\mathbb{RP}^\infty \times \cdots \times \mathbb{RP}^\infty;\mathbb{Z}/2\mathbb{Z})$, we have $w(\eta^n)$ unique. Thus by theorem 1.4.4, we obtain the results. $\square$

Moreover, with the help of the Grassmannians, we can prove the following tensor formula.

**Theorem 1.4.6.** *For $\xi$ an $m$-dimensional bundle and $\eta$ an $n$-dimensional bundle over $B$, the Stiefel-Whitney class $w(\xi\otimes\eta)$ is given by a universal formula*

$$w(\xi\otimes\eta) = p_{m,n}(w_1(\xi),\ldots,w_m(\xi),w_1(\eta),\ldots,w_n(\eta))$$

*Where $p_{m,n}$ can be characterized by the following identity:*

$$p_{m,n}(\sigma_1,\ldots,\sigma_m,\sigma_1',\ldots,\sigma_n') = \prod_{i=1}^m\prod_{j=1}^n(1+t_i+t_j')$$

*where $\sigma$ and $\sigma'$ are the elementary symmetric polynomials of $t_i$ and $t_j'$.*

*Proof.* It suffices to check the formula on the universal case: notice the operation of forming the tensor product bundle is characterized by a map $Gr_m(\mathbb{R})\times Gr_n(\mathbb{R}) \to Gr_{mn}(\mathbb{R})$. However, recall that we have a natural map inducing an injection on mod-2 cohomology $Gr_1(\mathbb{R})^k \to Gr_k(\mathbb{R})$, given by the classifying map of the $k$-fold direct sum of the canonical bundle. Therefore, we may consider the following diagram

$$\begin{array}{ccc} Gr_1(\mathbb{R})^{m+n} & \xrightarrow{\varphi_{m+n}} & Gr_m(\mathbb{R})\times Gr_n(\mathbb{R}) \\ {\scriptstyle\otimes}\downarrow & & \downarrow{\scriptstyle\otimes} \\ Gr_1(\mathbb{R})^{mn} & \xrightarrow[\varphi_{mn}]{} & Gr_{mn}(\mathbb{R}) \end{array}$$

where the left map is again the tensor map (since the tensor of vector bundles commutes with direct sums, the tensor product of direct sums of line bundles decomposes into direct sums of line bundles).
Thus it suffices to determine the map on the left.
We first consider the case where $m = n = 1$. Recall that $H^*(Gr_1(\mathbb{R})^2;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[t_1,t_2]$. Therefore $\otimes^* : H^*(Gr_1(\mathbb{R})) \to H^*(Gr_1(\mathbb{R})^2)$ is determined by $\otimes^*(t) = at_1 + bt_2$ for some constant $a$ and $b$. In this case, since the tensor product of a bundle with the 1-dimensional trivial bundle is itself, we have $a = b = 1$. Explicitly, we have proven that for two line bundles $\xi$ and $\eta$, $w_1(\xi\otimes\eta) = w_1(\xi) + w_1(\eta)$.
Then we consider the map $\otimes^* : H^*(Gr_1(\mathbb{R})^{mn}) \to H^*(Gr_1(\mathbb{R})^{m+n})$. The left is a polynomial ring with generators $t_{i,j}$, and the right is a polynomial ring with generators $s_i$ and $s_j'$. This map is given by $t_{i,j} \mapsto s_i + s_j'$, by the case of line bundles and the product formula of direct sum bundles.
Finally for the general case, recall that under the inclusion $H^*(Gr_n(\mathbb{R})) \to H^*(Gr_1(\mathbb{R})^n)$, the

image of $w_k(\eta^n)$ is $\sigma_k$. Therefore, we have

$$\begin{aligned}
&\otimes^* (w(\eta^{mn})) \\
=&\otimes^* (1 + w_1(\eta^{mn}) + \cdots + w_{mn}(\eta^{mn})) \\
=&\otimes^* (1 + \sigma_1 + \cdots + \sigma_{mn}) \\
=&\otimes^* (\prod(1 + t_{ij})) \\
=&\prod(1 + t_i + t'_j)
\end{aligned}$$

This implies the general case. □

*Remark* 5. The method used by passing from the case of $Gr_n$ to $Gr_1$ is called the splitting principle. We will repeatedly use this method throughout this part of the note.

## 1.5 Euler classes

Before we finally go into the non-trivial results of Stiefel-Whitney classes, we give a closer inspect to $\mathbb{Z}$-oriented vector bundles.

**Definition 1.5.1.** For an $\mathbb{Z}$-oriented $n$-plane bundle $\xi$, by theorem 1.3.7, the fundamental class $u \in H^n(E, E_0; \mathbb{Z})$ always exists. Then we define the Euler class of $\xi$, $e(\xi)$, to be image $(\pi^*)^{-1} i^*(u) \in H^n(B; \mathbb{Z})$, where $i^*$ is the canonical map $H^n(E, E_0; \mathbb{Z}) \to H^n(E, \mathbb{Z})$, and $\pi^*$ is the projection isomorphism $H^n(B; \mathbb{Z}) \to H^n(E; \mathbb{Z})$.

Unlike the $\mathbb{Z}/2\mathbb{Z}$ case, the sign of orientation matters.

**Corollary 1.5.2.** *For an orientation preserving bundle map $f : \xi \to \eta$ ($\xi$ and $\eta$ are always assumed to have the same dimension), $f^* w(\eta) = w(\xi)$.*
*If the choice of orientation of $\xi$ is inversed, then the Euler class changes sign.*

*Proof.* Direct by definition. □

**Corollary 1.5.3.** *For any odd dimensional bundle $\xi$, we have $2e(\xi) = 0$.*

*Proof.* By the Thom isomorphism (theorem 1.3.2) $\varphi(x) = p^*(x) \smile u$ maps $e(\xi)$ to the class $u \smile u$. Thus $u$ having odd dimension implies $2u \smile u = 0$, thus $2e(\xi) = 0$. □

Naturally, the Euler class relates to the top Stiefel-Whitney class.

**Proposition 1.5.4.** *The natural homomorphism $H^n(B; \mathbb{Z}) \to H^n(B; \mathbb{Z}/2\mathbb{Z})$ maps the Euler class (if it exists) to the top Stiefel-Whitney class.*

*Proof.* By the property of theorem 1.4.1, we have $Sq^n(u) = u \smile u$ since $u$ has dimension $n$. Now $e(\xi) = \varphi^{-1}(u \smile u)$, hence $e(\xi)$ maps to the class $\varphi^{-1}(u \smile u)$ in mod 2 cohomology, which is exactly $w_n(\xi)$. □

**Proposition 1.5.5.** *The Euler class of a product is given by $e(\xi \times \xi') = e(\xi) \times e(\xi')$. Similarly the Euler class of a Whitney sum is given by $e(\xi \oplus \xi') = e(\xi) \smile e(\xi')$.*

*Proof.* Suppose $\xi$ has dimension $m$ and $\xi'$ has dimension $n$. We first note that for the fundamental class we have $u(\xi \times \xi') = (-1)^{mn} u(\xi) \times u(\xi')$. Then considering the restriction homomorphism $H^{m+n}(E \times E', (E \times E')_0) \to H^{m+n}(E \times E') \simeq H^{m+n}(B \times B')$ gives $e(\xi \times \xi') = (-1)^{mn} e(\xi) \times e(\xi')$. Now, the Whitney sum case follows from pulling the product bundle along the diagonal $B \to B \times B$, hence $e(\xi \oplus \xi') = e(\xi) \smile e(\xi')$ by the definition of cross products. □

**Corollary 1.5.6.** *If a bundle has a nowhere vanishing cross section, then the Euler class is* 0.

*Proof.* $e(\xi) = e(\varepsilon^1) \smile e(\varepsilon^{1\perp})$. □

# Chapter 2

# Computations in Manifolds

## 2.1 Some common bundles of a manifold

We begin with the most canonical bundle associated to any manifold: the tangent bundle. But before we really start anything, we need some lemmas on normal bundles and tangent bundles.

**Lemma 2.1.1.** *The orientation of the tangent bundle $\tau_M$ is equivalent to an orientation on $M$.*

*Proof.* Recall the orientation of a manifold is a choice of a preferred generator $\mu_x \in H_n(M, M - x; A)$ compatible in the sense that $\mu_x$ and $\mu_y$ in some neighborhood $N$ homeomorphic to $\mathbb{R}^n$ comes from the same class in $H_n(M, M - N; A)$.
Similarly, the orientation of the bundle can be given by choosing a consistent generator $\mu_x \in H_n(\mathrm{T}_x M, \mathrm{T}_x M - 0; A)$ (under the natural pairing of the top cohomology and homology classes of spheres). But by the preceding corollary, we have $H_n(M, M - x; A) \simeq H_n(\mathrm{T}_x M, \mathrm{T}_x M - 0; A)$, which gives the results. □

**Lemma 2.1.2.** *The normal bundle $\nu^n$ associated with the diagonal embedding of $M$ into $M \times M$ is canonically isomorphic to the tangent bundle of $M$.*

*Proof.* We consider the tangent space $(u, v) \in \mathrm{T}_x M \times \mathrm{T}_x M \simeq$ is tangent to $\Delta(M)$ iff $u = v$ and normal to $\Delta(M)$ iff $u = -v$. Thus $(x, v) \mapsto ((x, x), (-v, v))$ gives an isomorphism of bundles. □

The next result arises from differential topology, and is crucial to some of the later results.

**Theorem 2.1.3.** *For an embedding of manifolds $f : M \to N$, there is an neighborhood of $M$, say $M' \subset N$, such that $M'$ is diffeomorphic to the total space of the normal bundle of $f(M)$ such that $x \in M$ has the image $f(x)$ the zero vector in the bundle under the identification $M' \simeq \nu_M$.*

**Corollary 2.1.4.** *If $f(M)$ is closed in $N$, then the cohomology ring $H^*(E, E_0; A)$ associated to the normal bundle of $M$ in $N$ is canonically isomorphic to $H^*(N, N - M; A)$. The same property applies to the homology.*

*Proof.* By excision, we have $H^*(N, N - M; A) \simeq H^*(M', M' - M; A) \simeq H^*(E, E_0; A)$. □

With the help of the tubular neighborhood lemma, we have a characterization of the Euler class of the normal bundle, and, consequently, the top Stiefel-Whitney class.

**Proposition 2.1.5.** *If $M$ is embedded as a closed submanifold of $N$ and the normal bundle $\nu^k$ is oriented0, the composition of the restriction homomorphisms*

$$H^k(E, E_0; \mathbb{Z}) \to H^k(N, N - M; \mathbb{Z}) \to H^k(N; \mathbb{Z}) \to H^k(M; \mathbb{Z})$$

*sends the fundamental class $u$ to the Euler class $e(\nu^k)$. Consequently, the modulo 2 fundamental class is send to the top Stiefel-Whitney class $w_k(\nu^k)$.*

*Proof.* This is by the definition of the Euler class, and the top Stiefel-Whitney class case is completely similar. □

**Corollary 2.1.6.** *If an $n$-dimensional manifold $M$ is embedded as a closed submanifold of $\mathbb{R}^{n+k}$, we have the top Stiefel-Whitney class $w_k(\nu^k) = 0$. In the oriented case $e(\nu^k) = 0$.*

*Proof.* $H^k(N; \Lambda) = 0$. □

## 2.2 The diagonal cohomology class

**Definition 2.2.1.** For a fixed coefficient ring $\Lambda$, we assume that an $n$-dimensional manifold $M$ is $\Lambda$ oriented. We define the cohomology class $u' \in H^n(M \times M, M \times M - \Delta(M); \Lambda)$ to be the fundamental class of the normal bundle of the diagonal embedding $\Delta : M \to M \times M$ (via the identification $H^n(M \times M, M \times M - \Delta(M); \Lambda) \simeq H^n(E, E_0; \Lambda)$ where $E$ is the total space of the normal bundle).
Note that here this is well-defined since when $M$ is $\Lambda$-oriented, the tangent bundle of $M$ is $\Lambda$-oriented, and under the isomorphism $\tau_M \simeq \nu^n$ in the diagonal embedding, the fundamental class of the normal bundle is well-defined.

We have a more explicit characterization of the fundamental class.

**Proposition 2.2.2.** *For an oriented $n$-dimensional manifold $M$, each $x \in M$, the cohomology class $H^n(M, M - x; \Lambda)$ has a preferred generator $u_x$ given by the perfect pairing $\langle u_x, \mu_x \rangle = 1$. Then, the diagonal cohomology class can be characterized as follows: under the inclusion $j_x$ : $(M, M - x) \hookrightarrow (M \times M, M \times M - \Delta(M))$ given by $y \mapsto (x, y)$, we have $j_x^*(u') = u_x$ for all $x$.*

*Proof.* By its definition, it admits the following unique characterization: for any $x$ and any small neighborhood $N$ of 0 in the tangent space $\mathrm{T}_x M$, consider the embedding $(N, N - 0) \to (M \times M, M \times M - \Delta(M))$, defined by the tubular neighborhood lemma. Then the induced cohomology morphism must map $d_M$ to the generator $u_x$. The rest follows from the embedding $(N, N - 0) \to (M, M - X)$. □

**Definition 2.2.3.** From the fundamental class, we define the diagonal cohomology class as the image of the restriction homomorphism $H^n(M \times M, M \times M - \Delta(M)) \to H^n(M \times M)$ of the fundamental class $u'$. We denote this class by $d_M$.

As its name, it behaves somehow like a diagonal.

**Lemma 2.2.4.** *For any class $a \in H^*(M)$, we have $(a \times 1) \smile d_M = (1 \times a) \smile d_M$.*

*Proof.* Let $N$ be a tubular neighborhood of the diagonal $\Delta(M)$ in $M \times M$, hence $\Delta(M)$ is a deformation retract of $N$. Now the projection $p_1, p_2 : M \times M \to M$ coincides on $\Delta(M)$, hence the restriction $p_1|_N$ and $p_2|_N$ are homotopic to each other. Thus we have $p_1^*(a) = a \times 1$ and $p_2^*(a) = 1 \times a$ has the same image under the restriction homomorphism $\iota^* : H^i(M \times M) \to H^i(N)$. This gives rise to the following commutative diagram:

$$\begin{tikzcd}
H^i(M \times M) \arrow[r] \arrow[d, "\smile u'"'] & H^i(N) \arrow[d, "\smile \iota^* u'"] \\
H^{i+n}(M \times M, M \times M - \Delta(M)) \arrow[r, "\simeq"'] & H^{i+n}(N, N - \Delta(M))
\end{tikzcd}$$

Thus we have $1 \times a$ and $a \times 1$ are equal when we pass to the bottom right cohomology group, hence by the bottom equivalence, we have $(1 \times a) \smile u' = (a \times 1) \smile u'$, and the restriction to $H^{i+n}(M \times M)$ gives $(1 \times a) \smile d_M = (a \times 1) \smile d_M$. □

**Definition 2.2.5.** We recall that we have a slant product operation $E^{p+q}(X\times Y;\Lambda)\otimes E_q(Y;\Lambda)\to E^p(X;\Lambda)$ by the following construction. Recall that all Eilenberg-Maclane spectra are ring spectrums. Then the information on the left is a morphism $f: X\wedge Y\to E$ and a morphism $g:\mathbb{S}\to E\wedge Y$. We may construct a map $X\to E$ as follows:

$$X \xrightarrow{\simeq} X\wedge\mathbb{S} \xrightarrow{1\wedge g} X\wedge E\wedge Y \xrightarrow{1\wedge c} X\wedge Y\wedge E \xrightarrow{f\wedge 1} E\wedge E \xrightarrow{\mu} E$$

where $\mu$ denotes the multiplication of the ring spectrum (for the explicit definition of smash products see part 2). Here we may substitute $E$ for any Eilenberg-Maclane spectra to get the desired definition.

**Lemma 2.2.6.** *For $a\in E^p(X)$, $b\in E^q(Y)$ and $c\in E_q(Y)$, we have $(a\times b)/c = a\langle b,c\rangle$.*

*Proof.* Now we have $a\in[X,E]_{-p}$, $b\in[Y,E]_{-q}$, then $a\times b$ is by definition induced by the following map:

$$X\wedge Y \xrightarrow{a\wedge b} E\wedge E \xrightarrow{\mu} E$$

Then $(a\times b)/c$ is induced by the composition

$$X \xrightarrow{\simeq} X\wedge\mathbb{S} \xrightarrow{1\wedge c} X\wedge E\wedge Y \xrightarrow{1\wedge c} X\wedge Y\wedge E \xrightarrow{a\times b\wedge 1} E\wedge E \xrightarrow{\mu} E$$

Then we notice that the morphism the $X$ part is always the identity until the final part. This implies the result by noticing the pairing $\langle -,-\rangle$ is a special case of the slant product. $\square$

By virtue of this definition and the above lemma, we have the following property of the diagonal class:

**Proposition 2.2.7.** *Suppose $M$ is compact and $\Lambda$-oriented so that we have the fundamental homology class $\mu$, then we have $d_M/\mu = 1\in H^0(M;\Lambda)$.*

*Proof.* We consider the image of $d_M/\mu$ under the restriction morphism $H^0(M)\to H^0(x)$ for any $x$. We consider the commutative diagram

$$\begin{tikzcd} H^n(M\times M) \arrow[r, "/\mu"] \arrow[d] & H^0(M) \arrow[d] \\ H^n(x\times M) \arrow[r, "/\mu"'] & H^0(x) \end{tikzcd}$$

Note that the left arrow sends $d_M$ to $1\times(i_x,1_M)^*(d_M)$, where $(i_x,1_M): M\to M\times M$ by $y\mapsto(x,y)$. Thus $i_x^*(d_M/\mu) = 1\times(i_x,1_M)^*(d_M)/\mu = 1\times\langle(i_x,1_M)^*(d_M),\mu\rangle\in H^0(x)$.
Then, recall that the fundamental homology class $\mu$ is characterized by the fact that the image of $\mu$ under the homomorphism $H^n(M)\to H^n(M,M-x)$ is the preferred generator $\mu_x$. We now prove $\langle(i_x,1_M)^*(d_M),\mu\rangle = 1$. We recall that $d_M$ comes from the fundamental class $u'\in H^n(M\times M, M\times M-\Delta(M))$. Now, by the following commutative square

$$\begin{tikzcd} H^n(M\times M, M\times M-\Delta(M)) \arrow[r] \arrow[d, "j_x"'] & H^n(M\times M) \arrow[d, "(i_x,1_M)^*"] \\ H^n(M,M-x) \arrow[r] & H^n(M) \end{tikzcd}$$

we have $(i_x,1_M)^*(d_M) = j_x^*(u')|_M$. But by proposition 2.2.2, we have $\langle j_x^*(u'),\mu_x\rangle = 1$, which is equal to $\langle j_x^*(u')|_M,\mu\rangle$. Thus we have $(d_M/\mu)|_x = 1$ for all $x$, thus $d_M/\mu = 1$. $\square$

**Theorem 2.2.8.** *Suppose $M$ is $\Lambda$-oriented, where $H^*(M)$ is free with basis $b_1, \dots, b_r \in H^*(M)$, then there exists a dual basis $b_1^\#, \dots, b_r^\# \in H^*(M)$ such that $\langle b_i \smile b_j^\#, \mu\rangle = \delta_{ij}$.*
*Moreover, we have the following formula for the diagonal class*

$$d_M = \sum_{i=1}^{r} (-1)^{\dim b_i} b_i \times b_i^\#$$

*Proof.* By the Kunneth formula, we have $d_M = b_1 \times c_1 + \cdots + b_r \times c_r$, where $\dim b_i + \dim c_i = n$. We apply the operation $/\mu$ to $(a \times 1) \smile d_M = (1 \times a) \smile d_M$, and then we have $((a \times 1) \smile d_M)/\mu = a \smile (d_M/\mu) = a$. On the right side, we substitute $\sum b_i \times c_i$ for $d_M$, then we have

$$((1 \times a) \smile d_M)/\mu = \sum (-1)^{\dim a \dim b_i}(b_i \times (a \smile c_i))/\mu = \sum (-1)^{\dim a \dim b_i} b_i \langle a \smile c_i\rangle$$

Thus the last expression is equal to $a$. Then since $a$ is an arbitrary cohomology class, we let it be $b_j$ and obtain that the coefficient $(-1)^{\dim a \dim b_i}\langle b_j \smile c_i, \mu\rangle$ must be $\delta_{ij}$. Thus we may let $b_i^\# = (-1)^{\dim b_i} c_i$, then we obtain the conclusion. □

**Corollary 2.2.9.** *If $M$ is compact oriented, then we have $\chi(M) = \langle e(\tau_M), \mu\rangle$.*

*Proof.* We have $e(\tau_M) = \Delta^*(d_M)$ by proposition 2.1.5. We may pass to rational coefficients and apply theorem 2.2.8, then we have

$$d_M = \sum (-1)^{\dim b_i} b_i \times b_i^\#$$

thus we have

$$e(\tau_M) = \sum (-1)^{\dim b_i} b_i \smile b_i^\#$$

By the characterization of the dual basis, we have the formula

$$\langle e(\tau_M), \mu\rangle = \sum (-1)^{\dim b_i} = \chi(M)$$

□

## 2.3 Wu class

**Definition 2.3.1.** We have an additive homomorphism $x \mapsto \langle Sq^k(x), \mu\rangle$ from $H^{n-k}(X; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$. By theorem 2.2.8, we have a cohomology class $v_k \in H^k(X; \mathbb{Z}/2\mathbb{Z})$ such that we have $\langle v_k \smile x, \mu\rangle = \langle Sq^k(x), \mu\rangle$ by letting $v_k$ be the sum of the dual basis of degree $k$. Actually, we may notice that we have $v_k \smile x = Sq^k(x)$.
We define the *total Wu class* $v \in H^\Pi(X)$ to be the formal sum $v = 1 + \cdots + v_n$. Clearly we have $\langle v \smile x, \mu\rangle = \langle Sq(x), \mu\rangle$.

**Lemma 2.3.2.** *We have $w_i(\tau_M) = Sq^i(d_M)/\mu$.*

*Proof.* Firstly by construction 1.4.2, we have $w_i(\tau_M) = \varphi^{-1} Sq^i(u)$, where $\varphi$ denotes the Thom isomorphism (in theorem 1.3.2), hence we have $Sq^i(u) = \varphi(w_i(\tau_M)) = (\pi^* w_i(\tau_M)) \smile u$. With the identification $H^*(E, E_0) \simeq H^*(M \times M, M \times M - \Delta(M))$, we have $Sq^i(u') = (w_i(\tau_M) \times 1) \smile u'$. Thus after passing to $H^*(M \times M)$, we have $Sq^i(d_M) = (w_i(\tau_M) \times 1) \smile d_M$.
Therefore, by applying the slant product $/\mu$ on both sides of the equation, we have $Sq^i(d_M)/\mu = ((w_i(\tau_M) \times 1) \smile d_M)/\mu = w_i \smile (d_M/\mu) = w_i$. □

**Theorem 2.3.3.** *For any manifold, the total Stiefel-Whitney class $w$ has a characterization $w(\tau_M) = Sq(v)$.*

*Proof.* We choose a basis $b_i$ for the modulo 2 cohomology class of $M$ and a dual basis $b_i^\#$ by theorem 2.2.8. Then we have $x = \sum b_i \langle x \smile b_i^\#, \mu\rangle$. Therefore by substituting $u$ for $x$ and applying the total Steenrod operation on both sides, we have

$$Sq(v) = \sum Sq(b_i)\langle Sq(b_i^\#), \mu\rangle = \sum (Sq(b_i) \times Sq(b_i^\#))/\mu = Sq(d_M)/\mu = w(M)$$

by the preceding lemma. □

# Chapter 3

# Chern Classes and Pontrajagin Classes

## 3.1 A quick review

**Definition 3.1.1.** An $n$-dimensional complex manifold $M$ is a second countable Hausdorff space where each point $x \in M$ such that it has a neighborhood homeomorphic to an open subset in $\mathbb{C}^n$, and the transitions between these open sets are given by holomorphic functions.

**Definition 3.1.2.** A complex bundle over a space $B$ is a projection $p : E \to B$, such that for each $b \in B$, the fibre $\pi^{-1}(b)$ can be identified with a complex vector space $V$. Moreover, it need to satiesfy the local triviality condition: for any $b \in B$, there exists a neighborhood $U$, such that the fibre $\pi^{-1}(U)$ is homeomorphic to $U \times V$. Moreover, the inclusion of each fibre $\pi^{-1}(b) \hookrightarrow \pi^{-1}(U)$ must be complex linear.

**Definition 3.1.3.** A complex structure on a real bundle $\xi$ is a continuous bundle map $J : E(\xi) \to E(\xi)$ such that $J$ is $\mathbb{R}$-linear on each fibre and satisfies the identity $J^2(v) = -v$ for every vector $v$ in the fibre.

**Corollary 3.1.4.** *If a $\mathbb{R}$-vector bundle admits a complex structure, then it has dimensions $2n$ for some $n$. Moreover, it can be regarded as a $n$-dimensional $\mathbb{C}$-vector bundle.*

**Definition 3.1.5.** A almost complex structure on a real manifold $M$ is a complex structure on its tangent bundle $\tau_M$.

## 3.2 Chern classes

**Lemma 3.2.1.** *For any complex vector bundle $\omega$, its underlying real vector bundle $\omega_{\mathbb{R}}$ has a preferred orientation.*

*Proof.* For any complex vector space $V$, we choose its basis $v_1, \dots, v_n$. Then $v_1, iv_1, \dots, v_n, iv_n$ determins a real basis of the underlying real vector space $V_{\mathbb{R}}$. Also, the real determinant of this set of basis is always positive. Hence a complex vector space has a preferred orientation. We may then pass easily to complex vector bundles. $\square$

**Corollary 3.2.2.** *For any complex vector bundle $\omega$ on $B$, its Euler class $e(\omega_{\mathbb{R}}) \in H^{2n}(B;\mathbb{Z})$ is well-defined.*

**Theorem 3.2.3.** *For a $n$-dimensional $\Lambda$-oriented bundle $\xi$, we have the following sequence:*

$$\cdots \longrightarrow H^{i-n}(B;\Lambda) \longrightarrow H^i(B;\Lambda) \longrightarrow H^i(E_0;\Lambda) \longrightarrow H^{i-n+1}(B;\Lambda) \longrightarrow \cdots$$

*Proof.* We already have the long exact sequence of cohomology

$$\cdots \longrightarrow H^i(E,E_0;\Lambda) \longrightarrow H^i(E;\Lambda) \longrightarrow H^i(E_0;\Lambda) \longrightarrow H^{i+1}(E,E_0;\Lambda)$$

The rest follows from theorem 1.3.2 by orientedness of $\xi$. □

We now construct the Chern class of a complex bundle.

**Construction 3.2.4.** For an $n$-dimensional complex bundle $\omega$ on $B$, we define its top Chern class $c_n(\omega)$ to be the Euler class $e(\omega_{\mathbb{R}})$. Then, we inductively define the lower classes.
Note that firstly, for a $n$-dimensional bundle $\omega$, we have an $n-1$-dimensional bundle $\omega_0$ on $E_0(\omega)$. Namely, we associate each point $v_x \in E(\omega)$ to its orthogonal complement in $F_x$. Then, we define $c_i(\omega) = (\pi_0^*)^{-1}(c_i(\omega_0))$. Note that $\pi_0^* : H^{2i}(B) \to H^{2i}(E_0)$ is an isomorphism for $i < n$ by theorem 3.2.3, so that the inverse image makes sense.
We then define the formal sum $c(\omega) = 1 + c_1(\omega) + \cdots + c_n(\omega)$ to be the total Chern class of $\omega$. Note that this is a unit.

**Lemma 3.2.5.** *For a bundle map $f_* : \omega \to \omega'$ over $f : B \to B'$, we have $c(\omega) = f^*(c(\omega'))$.*

*Proof.* Directly by the naturality of the Euler class and the naturality of the construction $\omega_0$. □

Similar to the real Grassmannian classifying real vector bundles, we have the complex Grassmannian classifies complex vector bundles.

**Theorem 3.2.6.** *For a paracompact topological space $X$, an $n$-dimensional $\mathbb{C}$-vector bundle $p : E \to X$ corresponds to a homotopy class of maps $f : X \to Gr_n(\mathbb{C})$. Moreover, this bundle is the pullback bundle of the canonical $n$-bundle $\eta^n$ over $Gr_n(\mathbb{C})$.*

We now give another characterization of Chern classes.

**Theorem 3.2.7.** *The cohomology ring $H^*(Gr_n(\mathbb{C});\mathbb{Z})$ is the polynomial ring over $\mathbb{Z}$ generated by the Chern classes $c_1(\eta^n), \dots, c_n(\eta^n)$.*

*Remark* 6. In the following parts, we have an analogous statement claiming that for suitable cohomology theory $E$, we have $E^*(Gr_n) \simeq (\pi_*(E))[[c_i]]$. Notice it is only in this part we adopt the convention that $H^* \simeq \bigoplus H^n$. In later parts, we will define $H^* \simeq \prod H^n$.

Before proving the theorem, we need some lemmas.

**Lemma 3.2.8.** *The cohomology ring $H^*(\mathbb{CP}^k;\mathbb{Z})$ is a truncated polynomial ring terminating in dimension $2k$, and generated by the first Chern class of the canoncial line bundle $c_1(\gamma_k^1)$.*

*Proof.* We use the Gysin sequence (theorem 3.2.3) on the canonical line bundle $\gamma_k^1$ of $\mathbb{CP}^k$. Using the definition of Chern classes, $c_1(\gamma_k^1) = e((\gamma_k^1)_{\mathbb{R}})$ since it is complex 1-dimensional, we have

$$\cdots \longrightarrow H^{i+1}(E_0;\mathbb{Z}) \longrightarrow H^i(\mathbb{CP}^k;\mathbb{Z}) \longrightarrow H^{i+2}(\mathbb{CP}^k;\mathbb{Z}) \longrightarrow H^{i+2}(E_0;\mathbb{Z}) \longrightarrow \cdots$$

Now that the space $E_0$ is the set of all pairs $(\ell, v)$ such that $\ell$ is a 1-dimensional subspace in $\mathbb{C}^{k+1}$, and $v \in \ell$ is non-zero. This can be identified with $\mathbb{C}^{k+1} - \{0\}$, hence is homotopic to $S^{2k+1}$. Thus for $0 \le i \le 2k-2$, we have $H^{i+2}(E_0) = 0$. Thus we have an isomorphism $\smile c_1 : H^i(\mathbb{CP}^k;\mathbb{Z}) \to H^{i+2}(\mathbb{CP}^k;\mathbb{Z})$. Beginning with $H^{-1}$ gives the result that all odd dimensional cohomology vanishes, and beginning with $H^0$ gives that all even dimensional cohomology is $\mathbb{Z}$. Finally, the isomorphism given by $\smile c_1$ gives that $c_1$ generates the cohomology freely. □

**Corollary 3.2.9.** *$H^*(Gr_1(\mathbb{C});\mathbb{Z})$ is the free polynomial over $\mathbb{Z}$ generated by $c_1(\eta^1)$.*

*Proof.* The above argument applies to the case $k = \infty$. □

Now we are ready to prove the theorem.

*Proof.* We prove by induction on $n$. Note that the case $n = 1$ is already solved in the preceding corollary.
Then, consider the map $f : E_0(\eta^n) \to Gr_{n-1}(\mathbb{C})$ defined by the following process: for a point $(V, v) \in E_0$, where $V$ is an $n$-subspace in $\mathbb{C}^\infty$ and $v$ a point in it, we define $f(V, v) = V \cap v^\perp$, which is an $n-1$-subspace in $\mathbb{C}^\infty$. We note here that $f$ can also be defined equivalently as the map classifying the orthogonal $n-1$-bundle on $E_0$.
We prove $f^* : H^*(Gr_{n-1}(\mathbb{C}); \mathbb{Z}) \to H^*(E_0; \mathbb{Z})$ is an isomorphism. We would need to take the orthogonal complement of a subspace, so we would first restrict $f$ to $f_N : E_0(\eta^n_N) \to Gr^N_{n-1}(\mathbb{C})$, where $Gr^n_m$ denotes the manifold of all $m$ subspace in $\mathbb{C}^n$, and $E_0(\eta^n_N)$ is the subbundle of $E_0(\eta^n)$ over $Gr^N_n(\mathbb{C})$. We note that we have a bundle $\omega_N^{N-n+1}$ over $Gr^N_{n-1}(\mathbb{C})$ where each fibre can be identified with the orthogonal complement of the $n-1$-subspace corresponding to a point in $Gr^N_{n-1}(\mathbb{C})$. This bundle can be identified with $f_N$ by definition. Using the Gysin sequence of this bundle, we have $f_N$ induces cohomology isomorphisms in dimension $\leq 2(N-n)$. Using cellular filtration we have $f$ induces cohomology isomorphism in all dimensions.
Then we may substitute $Gr_{n-1}(\mathbb{C})$ for $E_0$ in the Gysin sequence:

$$\cdots \longrightarrow H^i(Gr_n) \longrightarrow H^{i+2n}(Gr_n) \longrightarrow H^{i+2n}(Gr_{n-1}) \longrightarrow H^{i+1}(Gr_n) \longrightarrow \cdots$$

We prove the homomorphism $\lambda : H^{i+2n}(Gr_n) \to H^{i+2n}(Gr_{n-1})$ sends the Chern class $c_i(\eta^n)$ to $c_i(\eta^{n-1})$. By the equivalent definition of $f : E_0 \to Gr_{n-1}(\mathbb{C})$, we have $f^*(c_i(\eta^{n-1})) = c_i(\eta^n_0)$, and $\pi_0^*(c_i(\eta^n)) = c_i(\eta^n_0)$ is by construction 3.2.4.
It now remains to check the algebraic independence. We now apply the induction hypothesis. Since the map in the sequence $H^{i+2n}(Gr_n) \to H^{i+2n}(Gr_{n-1})$ is surjective, it splits into the following short exact sequences

$$0 \longrightarrow H^i(Gr_n) \xrightarrow{\smile c_n} H^{i+2n}(Gr_n) \xrightarrow{\lambda} H^{i+2n}(Gr_{n-1}) \longrightarrow 0$$

By this sequence, we prove the algebraic independence. For $x \in H^{i+2n}(Gr_n)$, $\lambda(x)$ can be uniquely expressed as a polynomial, say $\lambda(x) = p(c_1(\eta^{n-1}), \ldots, c_{n-1}(\eta^{n-1}))$, hence
$x - p(c_1(\eta^n), \ldots, c_{n-1}(\eta^n))$ is in the kernel of $\lambda$. Thus we have $x - p(c_1(\eta^n), \ldots, c_{n-1}(\eta^n)) = y \smile c_n(\eta^n)$ for some $y \in H^i(Gr_n)$. Now since the dimension of $y$ is lower then $x$, we apply another induction on the dimension and we may now suppose $y$ is uniquely a polynomial $q(c_1(\eta^n), \ldots, c_n(\eta^n))$. Now that $x = p(c_1(\eta^n), \ldots, c_{n-1}(\eta^n)) + c_n(\eta^n) q(c_1(\eta^n), \ldots, c_n(\eta^n))$, by applying $\lambda$ we see $p$ is unique, then $q$ is obviously unique. □

Similar to the product formula of the Stiefel-Whitney class, we have the following formula

**Theorem 3.2.10.** *For two bundles $\omega$, $\varphi$ over $B$, we have $c(\omega \oplus \varphi) = c(\omega)c(\varphi)$.*

**Lemma 3.2.11.** *There exists one and only polynomial $p_{m,n} = p_{m,n}(c_1, \ldots, c_m; c'_1, \ldots, c'_n)$ with the following identity $c(\omega \oplus \varphi) = p_{m,n}(c_1(\omega), \ldots, c_m(\omega); c_1(\varphi), \ldots, c_n(\varphi))$ for all $m$-bundle $\omega$ and $n$-bundle $\varphi$ over a common paracompact base space $B$.*

*Proof.* Note the space $Gr_m \times Gr_n$ classifies a pair consisting of an $m$-bundle and an $n$-bundle, where the universal bundle is constructed by the Whitney sum of the pullbacks of the two universal bundles, denoted by $\eta^m_1 \oplus \eta^n_2$, or the product bundle.
Now, since $Gr_n$ has free abelian cohomology, we have the Kunneth formula available $H^*(Gr_m) \otimes H^*(Gr_n) \simeq H^*(Gr_m \times Gr_n)$. Therefore $H^*(Gr_m \times Gr_n)$ is a polynomial ring over $\mathbb{Z}$ on the generators $c_i(\eta^m_1)$ and $c_j(\eta^n_2)$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Now for $\omega$ and $\varphi$ on $B$, it is classified by a map $f : B \to Gr_m \times Gr_n$. Hence we may take $p_{m,n}$ to be the expression of $c(\eta^m_1 \oplus \eta^n_2) = p_{m,n}(c_1(\omega), \ldots, c_n(\omega); c_1(\varphi), \ldots, c_n(\varphi))$. □

**Lemma 3.2.12.** *$p_{m,n}$ is given by $(1 + c_1 + \cdots + c_m)(1 + c'_1 + \cdots + c'_n)$.*

*Proof.* We proceed by induction on $m$ and $n$. Suppose inductively that $c(\eta_1^{m-1} \oplus \eta_2^n)$ is equal to $(1 + c_1 + \cdots + c_{m-1})(1 + c_1' + \cdots + c_n')$. Then for the bundle $\eta_1^{m-1} \oplus \varepsilon^1 \oplus \eta_2^n$ over $Gr_{m-1} \times Gr_n$, we have

$$c(\eta_1^{m-1} \oplus \varepsilon^1 \oplus \eta_2^n) = c(\eta_1^{m-1} \oplus \eta_2^n) = p_{m,n}(c_1(\eta_1^{m-1}), \ldots, c_m(\eta_1^{m-1}) = 0; c_1(\eta_2^n), \ldots, c_n(\eta_2^n))$$

By induction hypothesis, we have

$$p_{m,n}(c_1, \ldots, c_m = 0; c_1', \ldots, c_n') = (1 + c_1 + \cdots + c_{m-1})(1 + c_1' + \cdots + c_n')$$

By applying a similar induction on $n$, we have

$$p_{m,n}(c_1, \ldots, c_m; c_1', \ldots, c_n') = (1 + \cdots + c_m)(1 + \cdots + c_n') + u c_m c_n$$

By considering the highest dimension we have $u$ has dimension 0, or the total Chern class will have a element of dimension greater than $2m + 2n$. But $u = 0$ since the top Chern class can be identified with the Euler class, which satisfies the product formula. □

Now theorem 3.2.10 is a direct corollary.
With the help of this theorem, we may prove the following analogous statement in $Gr_n(\mathbb{C})$ as in $Gr_n(\mathbb{R})$.

**Theorem 3.2.13.** *Consider the product of the pullback bundles on $Gr_1(\mathbb{C})^n$, $\xi = \eta_1^1 \oplus \cdots \oplus \eta_n^1$. This bundle is classified by a map $\varphi : Gr_1(\mathbb{C})^n \to Gr_n(\mathbb{C})$. Then $\varphi^*(c_k(\eta^n)) = \sigma_k$, the $k$-th elementary symmetric polynomial of $c_1(\eta_1^1), \ldots, c_1(\eta_n^1)$.*

*Proof.* We have

$$c(\xi) = c(\eta_1^1) \cdots c(\eta_n^1) = \sum \sigma_k(c_1(\eta_1^1), \ldots, c_1(\eta_n^1))$$

The rest follows by lemma 3.2.5. □

Similar to the definition of Euler classes, some issues with orientations arises in Chern classes.

**Definition 3.2.14.** For a complex $n$-bundle $\omega$, we define its conjugate bundle $\bar{\omega}$ to be the bundle with the same underlying real bundle $\omega_{\mathbb{R}}$, and with the opposite complex structure, that is, $\bar{J}(x) = -J(x)$. Thus the identity map $f : E(\omega) \to E(\bar{\omega})$ is conjugate linear, explicitly $f(\lambda e) = \bar{\lambda} f(e)$.

**Proposition 3.2.15.** *We have $c(\bar{\omega}) = 1 - c_1(\omega) + \cdots + (-1)^n c_n(\omega)$.*

*Proof.* We prove by induction, hence it suffices to verify the top Chern class. But this follows from the vector space case by the following argument:
If $\{v_j, iv_j\}_{1 \le j \le n}$ is chosen as a basis for $F$, then $\{v_j, -iv_j\}_{1 \le j \le n}$ is the preferred basis for $\bar{F}$. Then the orientation is reversed for $n$ times, thus for the top class we have $c_n(\omega) = (-1)^n c_n(\bar{\omega})$. □

Finally, analogous to theorem 1.4.6, we have the following theorem.

**Theorem 3.2.16.** *For an $m$-dimensional complex vector bundle $\xi$ and an $n$-dimensional bundle $\eta$, we have $c(\xi \otimes \eta) = p_{m,n}(c_1(\xi), \ldots, c_m(\xi), c_1(\eta), \ldots, c_n(\eta))$, where $p_{m,n}$, as before, is given by*

$$p_{m,n}(\sigma_1, \ldots, \sigma_m, \sigma_1', \ldots, \sigma_n') = \prod (1 + t_i + t_j')$$

*Proof.* The same as theorem 1.4.6. □

**Definition 3.2.17.** We define the dual bundle of $\omega$, $\mathrm{Hom}_{\mathbb{C}}(\omega, \mathbb{C})$ to be the bundle over the same space with each fibre identified with $\mathrm{Hom}_{\mathbb{C}}(F_x, \mathbb{C})$.

**Corollary 3.2.18.** *If $\omega$ has a Hermitian metric, then* $\mathrm{Hom}_{\mathbb{C}}(\omega, \mathbb{C}) \simeq \bar{\omega}$.

*Proof.* Follows from the vector space case. □

We end this chapter again by calculating the Chern classes of the complex projective space.

**Proposition 3.2.19.** *The total Chern class $c_{\tau_{\mathbb{CP}^n}}$ is equal to $(1 - c_1(\gamma_1^1))^{n+1}$. Here, $\gamma_1^1$ is the canonical line bundle over $\mathbb{CP}^n$.*

*Proof.* Let $\gamma_1^1$ denote the canonical line bundle over $\mathbb{CP}^n$, and $\omega^n$ be its orthogonal complement in its tubular neighborhood. Again, we will show that the complex vector bundle $\mathrm{Hom}_{\mathbb{C}}(\gamma_1^1, \omega^n) \simeq \tau_{\mathbb{CP}^n}$. This is since a complex line $\ell$ through the origin of $\mathbb{C}^{n+1}$, the vector space $\mathrm{Hom}(\ell, \ell^{\perp})$ can be identified with the neighborhood of $\ell$ in $\mathbb{CP}^n$ where $\ell'$ near $\ell$ is regarded as the graph of the function.
Now adding the trivial bundle $\varepsilon^1$ to both sides of $\tau_{\mathbb{CP}^n} = \mathrm{Hom}(\gamma_1^1, \omega^1)$ we obtain

$$\tau_{\mathbb{CP}^n} \oplus \varepsilon^1 = \mathrm{Hom}(\gamma_1^1, \omega^1 \oplus \gamma_1^1) = \mathrm{Hom}(\gamma_1^1, \varepsilon^1 \oplus \cdots \oplus \varepsilon^1) = (\bar{\gamma}_1^1)^{n+1}$$

Hence we obtain the results immediately. □

## 3.3 Chern-Weil theory

Interestingly, characteristic classes also arises in complex geometry, described in terms of curvature forms in the De-Rham cohomology. This characterization is often referred to as Chern-Weil theory.

**Definition 3.3.1.** For a complex bundle $\zeta$, we define a connection on $\zeta$ is a $\mathbb{C}$-linear map

$$\nabla : C^\infty(\zeta) \to C^\infty(\tau^* \otimes \zeta)$$

satisfying the Leibniz formula $\nabla(fs) = \mathrm{d}f \otimes s + f\nabla(s)$. The image $\nabla(s)$ will be called the covariant derivative of $s$.

Intuitionally, the connection is very much like a derivative, and can be expressed locally.

**Theorem 3.3.2.** *Suppose that $\zeta$ is a trivial $m$-bundle on a chart $U \subset M$, then we have $\nabla e_j = \omega_j^i \otimes e_i = \Gamma_{j,k}^i \mathrm{d}u^k \otimes e_i$. Here $\omega_j^i$ is called the connection forms. Since $\omega_j^i$ is not well-defined in coordinate changes, these 1-forms are only defined locally.*
*Notice that the $m^2$ 1-forms $\omega_j^i$ form a (1,1)-tensor of 1-forms, hence is a matrix of 1-forms. We thus may regard $\omega$ as a section of $\tau_U^* \otimes \mathrm{End}(\zeta)$.*

*Remark* 7. The definition applies naturally to all principal $G$-bundles where $G$ is a Lie group. Notice that the case where $\zeta$ is a vector bundle is the special case where $G = \mathrm{GL}_n$, and $\mathrm{End}(\zeta) = \mathfrak{gl}(\zeta)$, the Lie algebra of $G$. This indicates that the connection of any principal $G$-bundle should be a section of $\tau_U^* \otimes \mathfrak{g}$.

**Definition 3.3.3.** We define the curvature form of a connection $\nabla$ to be the matrix of 2-forms

$$\Omega_j^i = d\omega_j^i + \omega_k^i \wedge \omega_j^k$$

**Proposition 3.3.4.** *For tangent vector fields $X$ and $Y$, $\Omega(X, Y) = R(X, Y) \in \Gamma(\mathrm{End}(\zeta))$, where $R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ is the Riemann curvature tensor.*

*Proof.* We first calculate $\nabla_X\nabla_Y - \nabla_Y\nabla_X$.

$$\begin{aligned}
&(\nabla_X\nabla_Y - \nabla_Y\nabla_X)(a^i e_i)\\
=&\nabla_X((Ya^j)e_j + a^k\langle Y, \omega_k^j\rangle e_j) - \nabla_Y((Xa^j)e_j + a^k\langle X, \omega_k^j\rangle e_j)\\
=&(X(Ya^l) + (Ya^m)\langle X, \omega_m^l\rangle + X(a^k\langle Y, \omega_k^l\rangle) + a^k\langle Y, \omega_k^m\rangle\langle X, \omega_m^l\rangle)e_l\\
-&(Y(Xa^l) + (Xa^m)\langle Y, \omega_m^l\rangle + Y(a^k\langle X, \omega_k^l\rangle) + a^k\langle X, \omega_k^m\rangle\langle Y, \omega_m^l\rangle)e_l\\
=&(X(Ya^l) - Y(Xa^l))e_l\\
+&(Y^j\frac{\partial a^m}{\partial u_j}X^k\Gamma_{m,k}^l - X^j\frac{\partial a^m}{\partial u_j}Y^k\Gamma_{m,k}^l)e_l\\
+&(X^j\frac{\partial a^k}{\partial u_j}Y^m\Gamma_{k,m}^l - Y^j\frac{\partial a^k}{\partial u_j}X^m\Gamma_{k,m}^l)e_l\\
+&(X^j a^k\frac{\partial Y^m}{\partial u_j}\Gamma_{k,m}^l - Y^j a^k\frac{\partial X^m}{\partial u_j}\Gamma_{k,m}^l)e_l\\
+&(X^j a^k Y^m\frac{\partial \Gamma_{k,m}^l}{\partial u_j} - Y^j a^k X^m\frac{\partial \Gamma_{k,m}^l}{\partial u_j})e_l\\
+&(a^k Y^j\Gamma_{k,j}^m X^n\Gamma_{k,n}^l - a^k X^j\Gamma_{k,j}^m Y^n\Gamma_{k,n}^l)
\end{aligned}$$

Notice the second and third line cancels out.
We then calculate $\nabla_{[X,Y]}$.

$$\begin{aligned}
&\nabla_{[X,Y]}(a^i e_i)\\
=&([X,Y]a^l)e_l + a^k\langle [X,Y], \omega_k^l\rangle e_l\\
=&(X(Ya^l) - Y(Xa^l))e_l + a^k\langle (X^j\frac{\partial Y^m}{\partial u_j} - Y^j\frac{\partial X^m}{\partial u_j})\frac{\partial}{\partial u_m}, \Gamma_{k,m}^l \mathrm{d}u_m\rangle e_l\\
=&(X(Ya^l) - Y(Xa^l))e_l + (X^j a^k\frac{\partial Y^m}{\partial u_j}\Gamma_{k,m}^l - Y^j a^k\frac{\partial X^m}{\partial u_j}\Gamma_{k,m}^l)e_l
\end{aligned}$$

which coincides the first line and the fourth line.
We finally calculate $\Omega(X,Y)$.

$$\begin{aligned}
&\Omega(X,Y)(a^i e_i)\\
=&(\mathrm{d}\omega_k^l(X,Y) + \omega_m^l\wedge\omega_k^m(X,Y))a^k e_l\\
=&\frac{\partial\Gamma_{k,m}^l}{\partial u_j}(X^jY^m - Y^jX^m)e_l + (\Gamma_{m,n}^l X^n\Gamma_{k,j}^m Y^j - \Gamma_{m,n}^l Y^n\Gamma_{k,j}^m X^j)e_l
\end{aligned}$$

which coincides with the fifth line and the sixth line. □

**Corollary 3.3.5.** *$\Omega$ is a globally defined matrix of 2-forms.*

*Proof.* $\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$ is obviously globally defined. □

Then, the curvature form satisfies the following Bianchi equality.

**Theorem 3.3.6.** *We have $\mathrm{d}\Omega_k^i = \Omega_j^i\wedge\omega_k^j - \omega_j^i\wedge\Omega_k^j$. We denote this by $\mathrm{d}\Omega = [\Omega, \omega]$.*

*Proof.*

$$
\begin{aligned}
&\mathrm{d}\Omega^i_k\\
=&\mathrm{d}(\mathrm{d}\omega^i_k+\omega^i_j\wedge\omega^j_k)\\
=&(\mathrm{d}\omega^i_j)\wedge\omega^j_k-\omega^i_j\wedge(\mathrm{d}\omega^j_k)\\
=&(\Omega^i_j-\omega^i_l\wedge\omega^l_j)\wedge\omega^j_k-\omega^i_j\wedge(\Omega^j_k-\omega^j_l\wedge\omega^l_k)\\
=&\Omega^i_j\wedge\omega^j_k-\omega^i_j\wedge\Omega^j_k\\
+&\omega^i_j\wedge\omega^j_l\wedge\omega^l_k-\omega^i_l\wedge\omega^l_j\wedge\omega^j_k
\end{aligned}
$$

and the final line vanishes. ☐

We then recall some basic definitions from linear algebra.

**Definition 3.3.7.** An invariant polynomial of $n\times n$-matrix is a polynomial $P:\mathbb{C}^{n\times n}\simeq M_{n\times n}(\mathbb{C})\to\mathbb{C}$, satisfying the following equation for all matrices $A$ and invertible matrices $T$

$$P(TAT^{-1})=P(A)$$

**Theorem 3.3.8.** *$P(A)$ is of the form $P(A)=p(\operatorname{Tr}A,\cdots,\operatorname{Tr}A^n)$.*

*Proof.* $P$ is a symmetric polynomial of the $n$ eigenvalues of $A$. ☐

**Corollary 3.3.9.** *The invariant polynomials are generated by the elementary symmetric polynomials of the eigenvalues, explicitly*

$$\sigma_k(A)=\sum_{1\le i_1<\cdots<i_k\le n}\lambda_{i_1}\cdots\lambda_{i_k}$$

Notice for any invariant polynomial $P$, we have a map $1\otimes P:\Gamma(\Omega^2\otimes\operatorname{End}(\zeta))\to\Gamma(\Omega^2)$.

**Proposition 3.3.10.** *$P(\Omega)\in\Omega^{2*}(M,\mathbb{C})$ is a closed form, where $\Omega$ is the curvature form.*

*Proof.* From theorem 3.3.8 we see that it suffices to prove $\mathrm{d}\Omega^k=0$. By theorem 3.3.6 and induction, we have $\mathrm{d}\Omega^k=[\Omega^k,\omega]$. Thus we have

$$\mathrm{d}\operatorname{Tr}\Omega^k=\operatorname{Tr}\mathrm{d}\Omega^k=\operatorname{Tr}[\Omega^k,\omega]=0$$

☐

**Proposition 3.3.11.** *The cohomology class $[P(\Omega)]\in H^{2*}(M,\mathbb{C})$ does not depend on the choice of the connection.*

*Proof.* Let $\nabla^0$ and $\nabla^1$ be two connections on $\zeta$. We consider the vector bundle $\zeta'=p\times 1:E\times\mathbb{R}\to M\times\mathbb{R}$, with connection $\nabla=t\tilde\nabla^1+(1-t)\tilde\nabla^0$, where $\tilde\nabla^i$ is defined by the pullback of the connection forms of $\nabla^i$ along $M\times\mathbb{R}\to M$.
Let $i_0,i_1:M\to M\times\mathbb{R}$ be the inclusion at respectively 0 and 1. We let $\Omega^0$ and $\Omega^1$ be the curvature form of $\nabla^0$ and $\nabla^1$, then we have

$$[P(\Omega^0)]=i_0^*[P(\Omega)]=i_1^*[P(\Omega)]=[P(\Omega^1)]$$

where the middle equality is since $i_0$ and $i_1$ are homotopic. ☐

Now, we have showed that for any invariant polynomial $P$, $[P(\Omega)]$ is a well-defined cohomology class. We will see that for certain invariant polynomial $P$, $[P(\Omega)]$ are exactly the Chern classes. Or more precisely, we have the following.

**Theorem 3.3.12.** *For a complex vector bundle $\zeta$ on a manifold $M$, we have*

$$c_k(\zeta) = \frac{1}{(2\pi i)^k}[\sigma_k(\Omega)]$$

Before the theorem, we need the following lemma.

**Lemma 3.3.13.** *Let $\zeta : E \to X$ be an $n$-dimensional complex vector bundle over a space $X$, then there exists a space equipped with a map $f : \tilde{X} \to X$ such that $f^*\zeta \simeq \ell_1 \oplus \cdots \oplus \ell_n$, where $\ell_i$ are complex line bundles over $\tilde{X}$. Moreover, the map $f^* : H^*(X) \to H^*(\tilde{X})$ is injective.*

*Proof.* We split off one line bundle at a time, and prove by induction. Consider the projective bundle $\mathbb{CP}^{n-1} \to P(\zeta) \xrightarrow{g} X$ corresponding to the complex vector bundle $\zeta$. Then $g^*\zeta$ has a canonical 1-dimensional subbundle $\ell_1$ with fibre over $(x, L)$ the line $L$. Then $g^*V = \ell_1 \oplus \ell_1^\perp$. Then, notice $\mathbb{CP}^{n-1}$ has the cohomology ring a truncated polynomial ring, hence is free in all dimensions. Thus $g^*$ is injective by theorem 1.3.3. We apply induction and immediately obtain the result. □

*Proof.* We now prove the theorem. We first prove the case for the first Chern class of a line bundle. Suppose $\zeta$ is a complex vector bundle over $M$, then suppose the classifying map of $\zeta$ is $\varphi : M \to \mathbb{CP}^\infty$. Since $E(\zeta)$ is a finite dimensional CW complex, by the cellular approximation theorem, we may homotope $\varphi$ to a map that factors through $\mathbb{CP}^N$. Moreover, we may assume $\varphi$ is smooth by some techniques in differential topology. Since $H^2(\mathbb{CP}^n;\mathbb{C}) = \mathbb{C}c_1(\gamma_1^1)$. Then let $\Omega$ be the canonical connection of $\gamma_1^1$. We have $[\Omega] = \alpha c_1(\gamma_1^1)$. Then $[\Omega(\zeta)] = \alpha\varphi^*(c_1(\gamma_1^1)) = \alpha c_1(\zeta)$. It then suffices to determin $\alpha$. We first focus on a 2-dimensional orientable Riemannian manifold $M$, and let $\tau_M$ be its tangent bundle. Then we have a canonical almost complex structure on $M$ given by rotating a vector in $\mathrm{T}_x$ by $\pi/2$ counterclockwise. We denote the Levi-Civita connection on $\tau_M$ by $\nabla$. We then take a chart and let $e_1$ satisfy $|e_1| \equiv 1$. Then let $e_2 = ie_1$. Now, since length is preserved under parallel shifting, we have $\nabla_X e_1 \perp e_1$ for all $X$. Thus $\nabla e_1 = \omega_1^2 \otimes e_2$. Then, let the complexification of $\nabla$, $\nabla^{\mathbb{C}}$ be defined as $\nabla^{\mathbb{C}}s = \nabla s$ for all sections $s$. This is $\mathbb{C}$-linear since $\nabla$ is compatible with the Riemannian structure. Finally, from $\omega_1^2 \otimes_{\mathbb{C}} e_2 = i\omega_1^2 \otimes e_1$ we see that the complex connection form $\omega^{\mathbb{C}} = i\omega_1^2$ (it is 1-dimensional). Thus the complex curvature form is $\Omega^{\mathbb{C}} = i\Omega_1^2$.
We then apply the above calculations to the case $M = S^2$, equipped with the standard Riemann tensor. On one hand, by the pairing definition of integrals, we have

$$\int_{S^2} \Omega^{\mathbb{C}} = \langle \alpha c_1(\tau_{S^2}), \mu_{S^2}\rangle = \alpha\langle e(\tau_{S^2}), \mu_{S^2}\rangle = 2\alpha$$

On the other hand, by the above arguments, we have

$$\int_U \Omega^{\mathbb{C}} = i\int_U \Omega_1^2 = i\int_U K\mathrm{d}S = i \cdot \mathrm{Vol}(U)$$

By taking the partition of unity, we have $2\alpha = i \cdot \mathrm{Vol}(S^2) = 4\pi i$, hence $\alpha = 2\pi i$.
Finally, we treat the general case. Let $\zeta = p : E \to M$ be a $m$-dimensional complex vector bundle, and let

$$P(A) = \sum_{k=1}^{m} \frac{1}{(2\pi i)^k}\sigma_k(A) = \det(I + \frac{A}{2\pi i})$$

By the theorem, we need to prove that this invariant polynomial corresponds to the total Chern class $c(\zeta)$. By lemma 3.3.13, it suffices to prove the case where $\zeta = \ell_1 \oplus \cdots \oplus \ell_m$. Then, we take a connection $\nabla^i$ on each line bundle $\ell_i$, and take the connection on $\zeta$ to be $\bigoplus \nabla^i$. In other words, we define the connection forms of $\nabla$ to be $\omega = \operatorname{diag}\{\omega^1, \ldots, \omega^n\}$. Hence we have

$$\begin{aligned}
&P(\Omega)\\
=&\det \operatorname{diag}\left(1+\frac{\Omega^1}{2\pi i}, \ldots, 1+\frac{\Omega^m}{2\pi i}\right)\\
=&P(\Omega^1)\cdots P(\Omega^m)\\
=&c(\ell^1)\cdots c(\ell^m) \text{ by the case for line bundles}\\
=&c(E)
\end{aligned}$$

□

As a corollary, we may define the Pontrajagin class using differential forms.

**Corollary 3.3.14.** *For a vector bundle $\zeta = p : E \to M$ with a connection $\nabla$ and curvature $\Omega$, then the $k$-th Pontrajagin class can be characterized as follows.*

$$p_k(E) = \frac{1}{(2\pi)^2 k}[\sigma_{2k}(\Omega)] \in H^{4k}(M; \mathbb{R})$$

*Proof.* Notice that a connection on $E$ induces a natural connection on $\mathbb{C} \otimes E$, with equal connection forms (one w.r.t $\mathbb{R}$ and another w.r.t $\mathbb{C}$). □

As a final remark, we can prove the Chern-Gauss-Bonnet formula with this result. Before this, we first recall the Pfaffian of a skew-symmetric matrix.

**Definition 3.3.15.** For an oriented $n$-dimensional inner product space, let $e_1, \ldots, e_n$ be an orthonormal basis compatible with the orientation. Then we let $T$ be the linear map $T : \bigwedge^* V \to \mathbb{R}$ given by $T(\bigwedge^{<n} V) = 0$ and $T(e_1 \wedge \cdots \wedge e_n) = 1$.
For an element $A \in V \wedge V$, we may regard it as an element of $V \wedge V^*$ through the natural isomorphism given by the inner product, and hence is a skew-symmetric matrix. We define the Pfaffian of $A$ to be

$$\operatorname{pf}(A) = T(\sum_{k=0}^{\infty} \frac{A^{\wedge k}}{k!})$$

**Proposition 3.3.16.** *We have $(\operatorname{pf}(A))^2 = \det A$.*

*Proof.* Directly check using the block diagonalization of $A$. □

Finally we recall that the curvature form of a metric is skew-symmetric. Then we have the following.

**Theorem 3.3.17.** *Let $\zeta = p : E \to M$ be a real vector bundle of dimension $2n$, then we have*

$$e(E) = [(\frac{-1}{2\pi})^n \operatorname{pf}(\Omega)] \in H^{2n}(M)$$

*In particular, we have*

$$\chi(M) = \frac{-1}{(2\pi)^n} \int_M \operatorname{pf}(\Omega)$$

*Proof.* The second assertion follows immediately from the first one. Notice $e(\zeta)^2 = p_n(\zeta)$, and $\operatorname{pf}(\Omega)^2 = \det(\Omega) = \sigma_{2n}(\Omega)$. Thus we have

$$e(\zeta) = \pm\left[\left(\frac{1}{2\pi}\right)^n \operatorname{pf}(\Omega)\right]$$

Now it suffices to determin the sign. Notice that we may again pass to the canonical bundle $\eta^{2n}$ of $Gr_{2n}(\mathbb{R}^N)$ as in the proof of theorem 3.3.12, so that the sign is independent of the choice of the manifold or vector bundles.
When $n = 1$, again by considering $S^2$ we have the sign a minus sign. For general $n$, we again use the splitting principle to reduce to the line bundle case. Therefore the sign is $(-1)^n$. □

## 3.4 Pontrajagin classes

**Definition 3.4.1.** For a real vector space $V$, we may obtain its complexification $V \otimes \mathbb{C} \simeq V \oplus iV$. Similarly, for a real vector bundle $\xi$ over some space $B$, we define its complexification to be the complex bundle $\xi \otimes \mathbb{C} \simeq \xi \oplus i\xi$ with the underlying real bundle $\xi \oplus \xi$ and the obvious complex structure.

**Lemma 3.4.2.** *For a real bundle $\xi$, we have $\xi \otimes \mathbb{C} \simeq \overline{\xi \otimes \mathbb{C}}$.*

*Proof.* We directly map $\xi \otimes \mathbb{C} \to \overline{\xi \otimes \mathbb{C}}$ by $1_\xi \oplus -1_\xi$. This is an orientation preserving isomorphism. □

We thus have the following condition.

**Corollary 3.4.3.** *$c_{2i+1}(\xi \otimes \mathbb{C})$ is of order 2.*

*Proof.* By proposition 3.2.15 and the above lemma. □

Therefore the following definition discarding the odd Chern classes makes sense.

**Definition 3.4.4.** For a real bundle $\xi$ over $B$, we define its $i$-th Pontrajagin class $p_i(\xi) \in H^{4i}(B;\mathbb{Z})$ to be the class $(-1)^i c_{2i}(\xi \otimes \mathbb{C})$. Then the total Pontrajagin class is defined as $p(\xi) = 1 + p_1(\xi) + \cdots + p_{\lfloor n/2 \rfloor}(\xi)$, where $\lfloor n/2 \rfloor$ is the greatest integer smaller than $n/2$.

**Corollary 3.4.5.** *Pontrajagin classes are natural w.r.t bundle maps. Moreover, $p(\xi \oplus \varepsilon^k) = p(\xi)$.*

*Proof.* Follows from lemma 3.2.5 and theorem 3.2.10. □

The product formula, however, does not hold for most general cases since we ignored the odd dimensional classes. We still have the weaker version below though.

**Corollary 3.4.6.** *For real bundles $\xi$ and $\eta$, we have $2(p(\xi \oplus \eta) - p(\xi)p(\eta)) = 0$.*

*Proof.* Every term in the difference between them contains an odd ordered Chern class. □

However, we do have the tensor product formula.

**Construction 3.4.7.** By theorem 3.2.16, we have $p_i(\xi \otimes \eta) = c_{2i}((\xi \otimes \eta) \otimes \mathbb{C}) = c_{2i}((\xi \otimes \mathbb{C}) \otimes (\eta \otimes \mathbb{C}))$. The rest follows from the tensor product formula.

**Lemma 3.4.8.** *For a complex bundle $\omega$, we have $\omega_\mathbb{R} \otimes \mathbb{C} \simeq \omega \oplus \bar{\omega}$.*

*Proof.* We consider a complex $n$-dimensional vector space $V$. Then this condition is equivalent to $V = W \oplus iW$ for some real $n$-dimensional vector space $W$ with complex structure $(x, y) \mapsto (-y, x)$. Therefore we have $V \oplus \bar{V} = W \oplus iW \oplus W \oplus -iW = V_\mathbb{R} \otimes \mathbb{C}$ □

**Corollary 3.4.9.** *For any $n$-dimensional complex vector bundle $\omega$ over $B$, the Chern classes $c_i(\omega)$ and the Pontrajagin classes $p_i(\omega_\mathbb{R})$ are related by the following formula*

$$1 - p_1 + \cdots + (-1)^n p_n = (1 - c_1 + \cdots + (-1)^n c_n)(1 + c_1 + \cdots + c_n)$$

*Proof.* Follows from the preceding lemma, theorem 3.2.10 and proposition 3.2.15. □

**Lemma 3.4.10.** *The real $2n$-plane bundle $(\xi \otimes \mathbb{C})_\mathbb{R}$ is isomorphic to $\xi \oplus \xi$ under an isomorphism which either preserves or reverses orientation according to whether $n(n-1)/2$ is even or odd.*

*Proof.* Direct by aligning the basis in the order $v_1, \ldots, v_n, iv_1, \ldots, iv_n$. □

**Corollary 3.4.11.** *If $\xi$ is an oriented $2k$-plane bundle, then the Pontrajagin class $p_k(\xi)$ is equal to the square of the Euler class $e(\xi)$.*

*Proof.* By definition, $p_k(\xi) = (-1)^k c_{2k}(\xi \otimes \mathbb{C}) = (-1)^k e((\xi \otimes \mathbb{C})_{\mathbb{R}})$. But by the preceding lemma and proposition 1.5.5, we have $e(\xi \oplus \xi) = (-1)^{2k(2k-1)/2} e(\xi)^2 = (-1)^k e(\xi)^2$. $\square$

We end this section with a simple application of the Pontrajagin classes.

**Theorem 3.4.12.** *There are no almost complex structures on the sphere $S^{4k}$.*

*Proof.* The proof is by contradiction. If an almost complex structure $J$ exists on $S^{4k}$, then by lemma 3.4.8 we have $\tau_{S^{4k}} \otimes \mathbb{C} \simeq \tau_{S^{4k}} \oplus \overline{\tau_{S^{4k}}}$, where the conjugation is given by the complex structure. Notice that all cohomology except for the 0-th and $4k$-th degree vanish, hence all Pontrajagin classes and Chern classes vanish for $\tau_{S^{4k}}$ except for the bottom and top term. Then we have

$$\begin{aligned}
&1 + (-1)^k p_k(\tau_{S^{4k}}) \\
=& c(\tau_{S^{4k}} \otimes \mathbb{C}) \\
=& c(\tau_{S^{4k}} \oplus \overline{\tau_{S^{4k}}}) \\
=& c(\tau_{S^{4k}}) c(\overline{\tau_{S^{4k}}}) \\
=& (1 + c_{2k}(\tau_{S^{4k}}))^2 \text{ (by proposition 3.2.15)} \\
=& 1 + 2c_{2k}(\tau_{S^{4k}}) \\
=& 1 + 2e(\tau_{S^{4k}}).
\end{aligned}$$

Now, by corollary 2.2.9, we have $p_k(\tau_{S^4{}_k})$ non-trivial. However, since the normal bundle of the inclusion $S^n \hookrightarrow \mathbb{R}^{n+1}$ is trivial, we have the Pontrajagin classes of $\tau_{S^{4k}}$ vanishes by corollary 3.4.5. $\square$

## 3.5 Oriented Grassmannians

Recall from definition 1.4.3 the definition of $Gr_n(\mathbb{R})$. We define in this section $\widetilde{Gr}_n(\mathbb{R})$, the oriented Grassmannian.

**Definition 3.5.1.** For positive integers $n$ and $k$, we define $\widetilde{Gr}_n(\mathbb{R}^{n+k})$ to be the quotient space $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^{n+k})/R'$, where $R'$ is the equivalence relation where $f = g$ iff $\operatorname{im} f = \operatorname{im} g$ and $f$ and $g$ has the same orientation. We define $\widetilde{Gr}_n(\mathbb{R}) := \widetilde{Gr}_n(\mathbb{R}^\infty) = \varinjlim_k \widetilde{Gr}_n(\mathbb{R}^{n+k})$. Likewise we have a canonical $n$-bundle on $\widetilde{Gr}_n$, which we will denote by $\tilde{\eta}_n$.

*Remark* 8. It is easy to see that we have a 2-sheeted covering space map $\widetilde{Gr}_n(\mathbb{R}) \to Gr_n(\mathbb{R})$. Then by connectedness of $\widetilde{Gr}_n$ and that $\pi_1(Gr_n(\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$, we see that $\widetilde{Gr}_n(\mathbb{R})$ is the universal covering space of $Gr_n(\mathbb{R})$. This implies that $\widetilde{Gr}_n(\mathbb{R})$ is equivalent to $BSO(n)$ (it is a classical result in algebraic topology that the universal cover of $BO(n)$ is $BSO(n)$), hence classifies oriented $n$-bundles.

Now, the $\mathbb{Z}/2\mathbb{Z}$ coefficient cohomology of $\widetilde{Gr}_n(\mathbb{R})$ is easy to compute.

**Theorem 3.5.2.** *$H^*(\widetilde{Gr}_n(\mathbb{R}); \mathbb{Z}/2\mathbb{Z})$ is the free polynomial algebra generated by the Stiefel-Whitney classes $w_2(\tilde{\eta}^n), \ldots, w_n(\tilde{\eta}^n)$.*

*Proof.* We first recall the Gysin sequence from theorem 3.2.3. Then notice that for any 2-sheeted covering $\tilde{B} \to B$ is a $O(1)$-principal bundle over $B$, hence is associated to a line bundle $\xi$ over $B$. Therefore by the Gysin sequence we have a long exact sequence

$$\cdots \longrightarrow H^{i-1}(Gr_n; \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^i(Gr_n; \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^i(\tilde{G}r_n; \mathbb{Z}/2\mathbb{Z}) \longrightarrow \cdots$$

where the map $H^{i-1} \to H^i$ is given by the cup product with the first Stiefel-Whitney class of the principal bundle $w_1(\xi)$, and the map $H^i \to H^i$ is given by the pullback along the projection.

Notice that by theorem 1.4.5, $H^1(Gr_n;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, hence $w_1(\xi)$ is either $w_1(\eta^n)$ or 0. But if $w_1(\xi) = 0$, then we will have a short exact sequence at the zeroth cohomology

$$0 \longrightarrow H^0(Gr_n) \longrightarrow H^0(\tilde{Gr}_n) \longrightarrow H^0(Gr_n) \longrightarrow 0$$

which would imply that $\tilde{Gr}_n$ is not connected, a contradiction. Therefore $w_1(\xi) = w_1(\eta^n)$, which implies the result immediately. □

The computation of $\mathbb{Z}$ coefficient cohomology is much more difficult, so we will not present the computation in this note (maybe I will find some reference on Bockstein spectral sequences later). Instead, we calculate the $\Lambda$ coefficient cohomology, where $\Lambda$ is an integral domain containing $1/2$ (e.g. $\mathbb{Z}[1/2]$).

**Theorem 3.5.3.** *If $\Lambda$ is an integral domain containing $1/2$, then the cohomology ring $H^*(\widetilde{Gr}_{2m+1}(\mathbb{R});\Lambda)$ is a polynomial ring generated by the Pontrajagin classes $p_1(\tilde{\eta}^{2m+1}),\dots,p_m(\tilde{\eta}^{2m+1})$. Similarly $H^*(\widetilde{Gr}_{2m}(\mathbb{R});\Lambda)$ is a polynomial ring generated by $p_1(\tilde{\eta}^{2m}),\dots,p_{m-1}(\tilde{\eta}^{2m})$ and $e(\tilde{\eta}^{2m})$.*
*In other words, $H^*(\widetilde{Gr}_n(\mathbb{R});\Lambda)$ is generated by the classes $p_1,\dots,p_{\lfloor n/2\rfloor}$ and $e$, with the possible relation that $e^2 = p_{n/2}$ when $n$ is even.*

*Proof.* The proof is by induction on $n$. When $n = 1$, notice that $Gr_1(\mathbb{R}) \simeq \mathbb{RP}^\infty$, hence $\widetilde{Gr}_1(\mathbb{R}) \simeq S^\infty$, which is contractible.
Similar to the complex case, we have $H^*(E_0(\eta^n)) \simeq H^*(\widetilde{Gr}_{n-1})$ by sending $(f,x) \in E_0$ to $\operatorname{im} f \cap x^\perp$ preserving orientation. We have hence a long exact sequence

$$\cdots \longrightarrow H^i(\widetilde{Gr}_n) \longrightarrow H^{i+n}(\widetilde{Gr}_n) \longrightarrow H^{i+n}(\widetilde{Gr}_{n-1}) \longrightarrow H^{i+1}(\widetilde{Gr}_{n+1})$$

where the first map is given by $\smile e$, and the second map $\lambda$ sends $p_k(\tilde{\eta}^n)$ to $p_k(\tilde{\eta}^{n-1})$, similar to the complex case.
If $n$ is even, then by the induction hypothesis, $\lambda$ is surjective, hence we have short exact sequences

$$0 \longrightarrow H^i(\widetilde{Gr}_n) \longrightarrow H^{i+n}(\widetilde{Gr}_n) \longrightarrow H^{i+n}(\widetilde{Gr}_{n-1}) \longrightarrow 0$$

Since the cohomology ring of $\widetilde{Gr}_{n-1}$ is a polynomial ring generated by $p_1,\dots,p_{n/2-1}$, we have obviously that $\widetilde{Gr}_n$ is a polynomial ring generated by $p_1,\dots,p_{n/2-1}$ and $e$.
If $n$ is odd, then $\smile e$ is the zero map, since $e$ is of torsion 2 and $\Lambda$ is integral and contains $1/2$. Hence we have short exact sequences

$$0 \longrightarrow H^j(\widetilde{Gr}_{2m+1}) \longrightarrow H^j(\widetilde{Gr}_{2m}) \longrightarrow H^{j-2m}(\widetilde{Gr}_{2m+1}) \longrightarrow 0$$

Hence we have $H^*(\widetilde{Gr}_{2m+1})$ a subring of $H^*(\widetilde{Gr}_{2m})$. If we denote the graded subring of $H^*(\widetilde{Gr}_{2m})$ generated by $p_1,\dots,p_m$ by $A^*$, then clearly by the definition of $\lambda$ we have $A^* \subseteq \operatorname{im}\lambda$. We must prove the equality holds.
By induction hypothesis, every element of $H^j(\widetilde{Gr}_{2m})$ can be written as a sum $a + ea'$, where $a \in A^j$ and $a' \in A^{j-2m}$ (here $e$ is the Euler class of $\tilde{\eta}^{2m}$ with $e^2 = p_m$). Hence we have $\operatorname{rank} H^j(\widetilde{Gr}_{2m}) = \operatorname{rank} A^j + \operatorname{rank} A^{j-2m}$. But by the short exact sequence we have $\operatorname{rank} H^j(\widetilde{Gr}_{2m}) = \operatorname{rank} H^j(\widetilde{Gr}_{2m+1}) + \operatorname{rank} H^{j-2m}(\widetilde{Gr}_{2m+1})$. Since $A^* \subseteq \operatorname{im}\lambda$, we have $\operatorname{rank} A^j \le \operatorname{rank} H^j(\widetilde{Gr}_{2m+1})$, but by the two equations above, the equality must holds, which implies $A^* = \operatorname{im}\lambda$. □

## 3.6 Chern numbers and Pontrajagin numbers

In this final section, we introduce a somewhat more combinatorial part of the theory.

**Definition 3.6.1.** For $n$ a positive integer, we define $I = i_1, \dots, i_r$ as an unordered $r$-tuple a partition of $n$ iff $i_1 + \cdots + i_r = n$. For two partitions $I = i_1, \dots, i_r$ and $J = j_1, \dots, j_s$ respectively of $n$ and $m$, we define $IJ = i_1, \dots, i_r, j_1, \dots, j_s$ is a partition of $m + n$. A refinement of $I$ is of the form $I_1 \dots I_r$, where $I_r$ is a partition of $i_r$.

**Definition 3.6.2.** For a compact $n$-dimensional complex manifold $K$, and $I$ a partition of $n$, we define the $I$-th Chern number $c_I[K] = c_{i_1} \cdots c_{i_r}[K]$ to be the integer $\langle c_{i_1}(\tau_K) \cdots c_{i_r}(\tau_K), \mu_K \rangle$, where both sides lies in the $2n$-th dimensional (co)homology.

*Remark* 9. Recall that $\mathbb{CP}^n$ has $c_i(\tau) = \binom{n+1}{i} a^i$, where $\langle a^n, \mu \rangle = 1$. Therefore we have

$$c_{i_1} \cdots c_{i_r}[\mathbb{CP}^n] = \binom{n+1}{i_1} \cdots \binom{n+1}{i_r}$$

**Definition 3.6.3.** For a smooth compact $4n$-dimensional manifold $M$, and $I$ a partition of $4n$, we can likewise define the Pontrajagin numbers.

We can likewise define the Stiefel-Whitney numbers, though we will not focus on the definition.

**Definition 3.6.4.** For a smooth manifold $M$, and $I$ a partition of $n$, we can also likewise define the Stiefel-Whitney numbers. The Stiefel-Whitney numbers lie in $\mathbb{Z}/2\mathbb{Z}$, unlike the Chern numbers and the Pontrajagin numbers.

We then introduce the perhaps most important object in combinatorics.

**Definition 3.6.5.** We let $\mathcal{S} \subset \mathbb{Z}[t_1, \dots, t_n]$ be the subring consisting of all symmetric polynomials. Recall that $\mathcal{S} = \mathbb{Z}[\sigma_1, \dots, \sigma_n]$, where $\sigma_i$ is the $i$-th elementary symmetric polynomial.

**Lemma 3.6.6.** *If we endow $\mathcal{S}$ with the natural grading with $t_i$ having degree 1, then $\mathcal{S}^k$ is generated additively by $\sigma_{i_1} \cdots \sigma_{i_r}$ for $I = i_1, \dots, i_r$ a partition of $k$.*

*Proof.* Obvious. □

**Lemma 3.6.7.** *The above basis could be written explicitly as*

$$\sum t_{a_1}^{i_1} \cdots t_{a_r}^{i_r}$$

*for $I = i_1, \dots, i_r$ a partition of $k$, and the summation is taken over all $k$-tuples.*

*Proof.* Obvious. □

**Definition 3.6.8.** For a partition $I$ of $k$, and $n \geq k$, then we define $s_I$ as the unique polynomial satisfying the following equation

$$s_I(\sigma_1, \dots, \sigma_n) = \sum t_{a_1}^{i_1} \cdots t_{a_r}^{i_r}$$

Notice the definition of $s_I$ extends to the case where $n \leq k$, in this case the $\sigma_i$ for $i \geq n$ are replaced by 0.

**Lemma 3.6.9.** *Notice that for $I$ a partition of $k$ and an $n$-dimensional complex bundle $\omega$ on $B$, we have $s_I(c(\omega)) := s_I(c_1, \dots, c_n) \in H^{2k}(B; \mathbb{Z})$. Then we have for $\omega$ and $\omega'$ on $B$*

$$s_I(c(\omega \oplus \omega')) = \sum_{JK=I} s_J(c(\omega)) s_K(c(\omega'))$$

*Proof.* By definition, it suffices to prove that in $\mathbb{Z}[t_1, \ldots, t_m, s_1, \ldots, s_n]$ and a partition of $m+n$, $I$, we have (here $x_i$ is a substitution for $t_i$ and $s_j$)

$$\sum x_{a_1}^{i_1} \cdots x_{a_r}^{i_r} = \sum_{JK=I} \left(t_{b_1}^{j_1} \cdots t_{b_p}^{j_p}\right) \left(s_{c_1}^{k_1} \cdots s_{c_q}^{k_q}\right)$$

Which is directly by definition. □

**Corollary 3.6.10.** $s_k(c(\omega \oplus \omega')) = s_k(c(\omega)) + s_k(c(\omega'))$

**Definition 3.6.11.** According to the above defitions, for $I$ a partition of $n$, we define for an $n$-dimensional complex manifold $s_I(c)[K]$ to be the characteristic number $\langle s_I(c(\tau_K)), \mu_K \rangle$.

**Corollary 3.6.12.** *For complex manifolds $K$ and $L$, we have the following identity.*

$$s_I[K \times L] = \sum_{I_1 I_2 = I} s_{I_1}[K] s_{I_2}[L]$$

*Proof.* This is direct after noticing $\tau_{K\times L} = \pi_1^*(\tau_K) \oplus \pi_2^*(\tau_L)$. □

**Corollary 3.6.13.** *For any product $K \times L$ with $K$ $n$-dimensional and $L$ $m$-dimensional, we have $s_{m+n}(K \times L) = 0$.*

*Proof.* $m+n$ admits no non-trivial decomposition into juxtapositions. □

**Corollary 3.6.14.** *$\mathbb{CP}^n$ is not a product of two manifolds.*

*Proof.* It is obvious that $s_n[\mathbb{CP}^n] = n+1 \neq 0$. □

Finally, we prove that, in some sense, the Chern numbers are linearly independent.

**Theorem 3.6.15.** *Let $K^i$ be an $i$-dimensional compact complex manifold with $s_i(c)[K^i] \neq 0$. Then the $p(n) \times p(n)$ matrix ($p(n)$ is the number of partitions of $n$)*

$$\left[c_{i_1} \ldots c_{i_r}[K^{j_1} \times \cdots \times K^{j_s}]\right]$$

*is non-singular. Here $I = i_1, \ldots, i_r$ and $J = j_1, \ldots, j_s$ runs through all partitions of $n$.*

*Proof.* By the definition of $s_I(\sigma_1, \ldots, \sigma_n)$, we see that they form another basis of $\mathcal{S}^n$, hence it suffices to verify that the matrix

$$\left[s_I(c)[K^{j_1} \times \cdots \times K^{j_s}]\right]$$

is non-singular. As an immediate generalization of corollary 3.6.12, we have

$$s_I(c)[K^{j_1} \times \cdots \times K^{j_s}] = \sum_{I_1 \ldots I_s = I} s_{I_1}[K^{j_1}] \cdots s_{I_s}[K^{j_s}]$$

Note that for the product to be non-zero, $I_k$ must be a partition of $j_k$. Thus the number $s_I(c)[K^{j_1} \times \cdots \times K^{j_s}]$ is non-zero only if $I$ is a refinement of the partition $J = j_1, \ldots, j_s$. Thus the matrix, under a suitable order, is triangular, with diagonal $s_{i_1,\ldots,i_r}(c)[K^{i_1} \times \cdots \times K^{i_r}] = s_{i_1}(c)[K^{i_1}] \cdots s_{i_r}(c)[K^{i_r}] \neq 0$. Thus the matrix is non-singular. □

We have an analogous theorem for Pontrajagin classes.

**Theorem 3.6.16.** *Let $M^4, \ldots, M^{4n}$ be oriented manifolds with $s_k(p)[M^{4k}] \neq 0$, then the $p(n) \times p(n)$ matrix*

$$p_{i_1} \cdots p_{i_r}[M^{4j_1} \times \cdots \times M^{4j_s}]$$

*is non-singular.*

**Definition 3.6.17.** For an $n$-dimensional complex vector bundle $\omega$ on $M$, we define its Chern character to be the formal sum

$$\operatorname{Ch}(\omega) = n + \sum_{k=1}^{\infty} \frac{s_k(c(\omega))}{k!} \in H^{\Pi}(M;\mathbb{Z})$$

**Theorem 3.6.18.** *We have* $\operatorname{Ch}(\omega \oplus \omega') = \operatorname{Ch}(\omega) + \operatorname{Ch}(\omega')$.

*Proof.* This is obvious by corollary 3.6.10. □

**Theorem 3.6.19.** *We have* $\operatorname{Ch}(\omega \otimes \omega') = \operatorname{Ch}(\omega) + \operatorname{Ch}(\omega')$.

*Proof.* By lemma 3.3.13, it suffices to consider the case where both $\omega$ and $\omega'$ are direct sums of line bundles. However, by the additivity above, it suffices to treat the case where both $\omega$ and $\omega'$ are line bundles. In this case, $c(\omega) = 1 + c_1(\omega)$, and $\operatorname{Ch}(\omega) = \exp(c_1(\omega))$. Thus we obviously have $\operatorname{Ch}(\omega)\operatorname{Ch}(\omega') = \exp(c_1(\omega))\exp(c_1(\omega')) = \exp(c_1(\omega) + c_1(\omega')) = \exp(c_1(\omega \otimes \omega'))$. □

As in Chern-Weil theory, we have a characterization of the Chern character by the curvature tensor.

**Theorem 3.6.20.** *For a complex vector bundle $\zeta$ on a manifold $M$ with connection $\nabla$ and curvature $\Omega$, we have*

$$\operatorname{Ch}(\zeta) = [\operatorname{Tr}\exp(\frac{\Omega}{2\pi i})]$$

*Proof.* Recall from theorem 3.3.12 that we have

$$c_k(\zeta) = \frac{1}{(2\pi i)^k}[\sigma_k(\Omega)]$$

This implies that the eigenvalues of $\Omega$ (as a 2-form) divided by $2\pi i$, say $\lambda_1, \ldots, \lambda_n$, acts as the generators of $H^*(M;\mathbb{C}) = \mathbb{C}[[\lambda_1], \ldots, [\lambda_n]]$. Then

$$\operatorname{Ch}(\zeta) = \sum_{k=0}^{\infty} \frac{\lambda_1^k + \cdots + \lambda_n^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} \operatorname{Tr}\left(\frac{\Omega}{2\pi i}\right)^k = \operatorname{Tr}\exp(\frac{\Omega}{2\pi i})$$

□

# Chapter 4

# A little bit more on Pontrajagin classes

## 4.1 Cobordism rings

In this section, we make use of the Stiefel-Whitney numbers and the Pontrajagin numbers to develop some elementary theory on Cobordism rings. In this section, all homology and cohomology of $BO$ and $MO$ are assumed to have $\mathbb{Z}/2\mathbb{Z}$ coefficients. We shall very frequently use notations of stable homotopy theory in this section, which readers can refer to in the second part of this note.

**Definition 4.1.1.** We define $\mathfrak{N}_*$ to be the graded ring, with $\mathfrak{N}_m$ consisting of all unoriented cobordism class of $m$-dimensional compact manifolds ($M$ and $N$ are said to belong to the same cobordism class iff there is some $m+1$-dimensional manifold with boundary $P$ such that $\partial P = M \coprod N$), with addition the disjoint union and multiplication the cartesian product.
We similarly define $\Omega_*$ to be the graded ring with $\Omega_m$ consisting of all oriented cobordism class of $m$-dimensional compact manifolds (that is, the set of all manifolds equipped with an orientation form, and cobordisms are asked to respect the orientation of the boundary of the manifold).

**Definition 4.1.2.** We define $MO = \varinjlim MO(n)$, where $MO(n)$ is the Thom space of $\eta^n$, the universal bundle. We likewise define $MSO$ and $MU$.

We first prove the following theorem fundamental to cobordism theory.

**Theorem 4.1.3.** *We have* $\mathfrak{N}_n = \varinjlim_k \pi_{n+k}(MO(k))$

*Remark* 10. Notice that if we let $MO$ be the spectrum with the $k$-th space $MO(k)$, then the right hand side is the $n$-th homotopy group of $MO$, $\pi_*(MO)$. Also, recall that $MO$ is a ring spectra, hence $\pi_*(MO)$ is a graded ring. By construction, we would have $\mathfrak{N}_* \simeq \pi_*(MO)$ isomorphic as a ring. Notice also that the mod 2 homology $H_*(MO)$ is also a graded ring.

Before the proof, we need some simple facts in differential topology.

**Definition 4.1.4.** For a smooth map $f : M \to N$, and a submanifold $X \subset N$. We say $f$ is transverse to $X$, denoted by $f \pitchfork X$, if for every $x \in f^{-1}(X)$, we have $\mathrm{d}f_x(\mathrm{T}_x M) + \mathrm{T}_{f(x)} X = \mathrm{T}_{f(x)} N$.

**Theorem 4.1.5.** *Let $f : M \to N$ be a smooth map and $X$ be a submanifold of $N$. If $f \pitchfork X$, then $f^{-1}(X)$ is a submanifold of $M$.*

**Theorem 4.1.6.** *Let $f : M \to N$ be a smooth map and $X$ be a submanifold of $N$. Then there exists a smooth map $g : M \to N$, $g \pitchfork X$, and $f$ is homotopic to $g$.*

*Proof.* The proof will consist of three steps. The first one is to construct a well-defined map of abelian groups $\alpha : \mathfrak{N}_n \to \pi_n(MO)$. The second one is to construct $\beta : \pi_n(MO) \to \mathfrak{N}_n$. Finally we will verify that $\beta$ is indeed the inverse of $\alpha$.

Now, given a cobordism class $[M] \in \mathfrak{N}_n$, let $M \in [M]$ be a representative. By the Whitney embedding theorem, we have an embedding $e : M \to \mathbb{R}^{n+k}$. Then, by theorem 2.1.3, we suppose there is a tubular neighborhood $T$ of $e(M)$. Now, obviously $\bar{T}/\partial\bar{T} \simeq \mathrm{Th}(\nu_M)$, where $\nu_M$ is the $k$-dimensional normal bundle of $e$. Then, let $S^{n+k}$ be the one-point compactification of the $\mathbb{R}^{n+k}$ above, so that we have a map $\eta : S^{n+k} \to \bar{T}/\partial\bar{T} \simeq \mathrm{Th}(\nu_M)$, often called the Thom collapse map. Therefore we already have an element $\eta \in \pi_{n+k}(\mathrm{Th}(\nu_M))$. We now consider the classifying map of $\nu_M$, $\varphi$. From the bundle map $\nu_M \to \eta_k$ we have a map $\mathrm{Th}(\nu_M) \to \mathrm{Th}(\eta_k)$, therefore $\varphi_*(\eta)$ is an element of $\pi_{n+k}(MO(k)) \subseteq \pi_n(MO)$.

Now, we prove $\alpha$ is independent of the choice of the cobordism representative $M$. It suffices to show that if $M$ is null-bordant, then $\alpha[M]$ is null-homotopic. Let $M = \partial W$. Then the inclusion $T \hookrightarrow \mathbb{R}^{n+k} \hookrightarrow S^{n+k}$ extends to a tubular neighborhood of $W$, $L$, to the disc $D^{n+k+1}$ (especially note here that the normal bundle $\nu_W$ is also of dimension $k$). Notice then that the two collapse maps agree, so that we have the following commutative square.

$$\begin{tikzcd} S^{n+k} \arrow[r] \arrow[d] & \mathrm{Th}(\nu_M) \arrow[d] \\ D^{n+k+1} \arrow[r] & \mathrm{Th}(\nu_W) \end{tikzcd}$$

Also, since the $\nu_W$ restricts to $\nu_M$ on $M$ (directly consider the tubular neighborhood, as its homotopic to the total space $\nu$ of the normal bundle), the classifying map of $\nu_M$ factors through the classifying map of $\nu_W$, say $\psi$, by the inclusion $\iota : M \hookrightarrow W$. Therefore we have $\varphi_* = \psi_* \circ \iota_* : \nu_M \to \nu_W \to \eta^k$, and the map commutes at the level of Thom spaces. Therefore, by construction, the map $\varphi_* \circ \eta$ is null-homotopic through the inclusion into $D^{n+k+1}$.

Then we verify that $\alpha[M \coprod N] = \alpha[M] + \alpha[N]$. We let $i : M \hookrightarrow \mathbb{R}^{n+k}$ and $j : N \hookrightarrow \mathbb{R}^{n+k}$ be two embeddings. Since both $M$ and $N$ are compact, we may suppose that their tubular neighborhoods lie in the distinct hemisphere of $S^{n+k}$. We notice that by definition we have $\mathrm{Th}(\nu_M) \vee \mathrm{Th}(\nu_N) \simeq \mathrm{Th}(\nu_{M \coprod N})$. Then by the construction above, the map $S^{n+k} \to \mathrm{Th}(\nu_{M \coprod N})$ factors as below.

$$S^{n+k} \to S^{n+k} \vee S^{n+k} \to \mathrm{Th}(\nu_M) \vee \mathrm{Th}(\nu_N) \simeq \mathrm{Th}(\nu_{M \coprod N})$$

After composing with the map $\mathrm{Th}(\nu) \to MO(k)$, we obtain the result directly.

We then construct $\beta$. Suppose we have some $[\varphi] \in \pi_n(MO)$, and a representative $\varphi : S^{n+k} \to MO(k)$. Since $S^{n+k}$ is compact, $\varphi$ factors through $\mathrm{Th}(\eta^k_{k+m})$, the Thom space of the canonical bundle of $Gr_k(\mathbb{R}^{k+m})$, since $Gr_k(\mathbb{R}) = \varinjlim Gr_k(\mathbb{R}^{k+m})$. Now let $\xi : Gr_k(\mathbb{R}^{k+m}) \to \mathrm{Th}(\eta^k_{k+m})$ denote the zero section of the bundle. We view $\xi(Gr_k(\mathbb{R}^{k+m}))$ as an embedded submanifold of $\mathrm{Th}(\eta^k_{k+m})$ of codimension $K$. By theorem 4.1.6, we have a map $f : S^{n+k} \to \mathrm{Th}(\eta^k_{k+m})$ such that $f$ is smooth, $f \simeq \varphi$, and $f \pitchfork \xi(Gr_k(\mathbb{R}^{k+m}))$. Thus let $X = f^{-1}(\xi(Gr_k(\mathbb{R}^{k+m})))$. By theorem 4.1.5, $X$ is a submanifold of $S^{n+k}$ of codimension $k$, hence an $n$-dimensional manifold. Notice $X$ is compact since $\xi(Gr_k(\mathbb{R}^{k+m}))$ is closed. We define $X = \beta[\varphi]$.

We then verify that $\beta$ is well-defined. Suppose we are given homotopic maps $\varphi \simeq \psi \in \pi_n(MO)$. By definition WLOG $\varphi$ and $\psi$ factors through $\mathrm{Th}(\eta^k_{k+m})$, as well as the homotopy between $\varphi$ and $\psi$, say $h$. Then we may find a smooth map $\mu : I \times S^{n+k} \to \mathrm{Th}(\eta^k_{k+m})$ homotopic to $h$ relative $\partial I \times S^{n+k}$, and transverse to $\xi(Gr_k(\mathbb{R}^{k+m}))$. By theorem 4.1.5 again, we have $\mu^{-1}(\xi(Gr_k(\mathbb{R}^{k+m})))$ a cobordism between the manifolds determined by $\varphi$ and $\psi$.

Finally we verify that $\beta$ is an inverse of $\alpha$. We first consider $\beta \circ \alpha$. Let $M$ be a arbitrary $n$-dimensional compact manifold. Recall $\alpha[M]$ is given by $f\eta : S^{n+k} \to \mathrm{Th}(\nu_M) \to MO(k)$. Notice that by compactness again, the latter map factors through some $\mathrm{Th}(\eta^k_{k+m})$. By construction, the collapse map is constant on the tubular neighborhood $T$, and the second map sends only $M$ to the zero section in the construction of $\beta$. Thus $\beta\alpha[M] = [M]$. We then consider the inverse. Let $[\varphi] : S^{n+k} \to MO(k)$, and suppose the map factors through $\mathrm{Th}(\eta^k_{k+m})$ for some $m$, and is transverse to the zero section. Let $M$ be the result of the construction, so that $[M] = \beta[\varphi]$. Notice $\mathrm{Th}(\eta^k_{k+m}) - \infty$ is a tubular neighborhood around the image of the zero section, which pulls back to a tubular neighborhood of $M$ in the $S^{n+k}$. This implies that we have the map $\varphi : S^{n+k} \to \mathrm{Th}(\eta^k_{k+m})$ factors through $\mathrm{Th}(\nu_M)$ by the collapse map. Therefore we have $\varphi = \alpha[M] = \alpha\beta[\varphi]$. □

**Corollary 4.1.7.** *We have $\Omega_n \simeq \pi_n(MSO)$.*

*Proof.* The proof is analogous. □

With the help of theorem 4.1.3, we may begin to analyze the structure of the cobordism groups. Before this, we first recall from theorem 1.4.1 the definitions and properties of the Steenrod algebra.

**Definition 4.1.8.** We define $\mathcal{A}$ to be the graded associative (not commutative) $\mathbb{Z}/2\mathbb{Z}$-algebra with grade $\mathcal{A}^r$ consisting of all stable cohomology operations on $H^*(-, \mathbb{Z}/2\mathbb{Z})$ of degree $r$, with addition given by addition and multiplication given by composition.
By definition, the mod 2 cohomology of $X$ is $[X, \Sigma^n H\mathbb{Z}/2\mathbb{Z}]$, hence by Yoneda's theorem, the degree $r$ operations are precisely $[H\mathbb{Z}/2\mathbb{Z}, \Sigma^r H\mathbb{Z}/2\mathbb{Z}] \simeq H^r(H\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$.

**Theorem 4.1.9.** *$\mathcal{A}$ is isomorphic to the $\mathbb{Z}/2\mathbb{Z}$ algebra generated by $Sq^i$ taking quotients with the Adem relations (recall from theorem 1.4.1 the definition of Adem relations).*

**Construction 4.1.10.** Notice that $\mathcal{A}$ can be given a coproduct structure $\mu : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, explicitly by

$$Sq^k \mapsto \sum_{i+j=k} Sq^i \otimes Sq^j$$

This, along with checking other basic properties, endows $\mathcal{A}$ the structure of a hopf algebra.
Then, the dual algebra $\mathcal{A}_* = \mathrm{Hom}(\mathcal{A}, \mathbb{Z}/2\mathbb{Z})$ inherits a natural commutative hopf algebra structure (whose algebra structure is inherited from the coalgebra structure of $\mathcal{A}$).

**Theorem 4.1.11.** *We define $Sq^{I_r} = Sq^{2^{r-1}} Sq^{2^{r-2}} \cdots Sq^1$. Then, $\mathcal{A}_* \simeq \mathbb{Z}/2\mathbb{Z}[\xi_r]$, where $\xi_r$ is dual to $Sq^{I_r}$. Moreover, the coalgebra structure of $\mathcal{A}_*$ is given by*

$$\psi : \xi_r \mapsto \sum_{i+j=r} \xi_i^{2^j} \otimes \xi_j$$

**Theorem 4.1.12.** *For an oriented $n$-bundle $\xi$ over $B$, we have $H_{*+n}(E, E_0) \simeq H_*(E)$. Moreover, the isomorphism is given by $\alpha \mapsto \alpha/u$, where $u$ is the fundamental class of the orientation.*

*Proof.* It suffices to prove for the case of the trivial bundle. The rest follows from a Mayer-Vietoris argument and the fact that $H_*(B) \simeq \varinjlim_K H_*(K)$, where $K$ runs through all compact subsets of $B$. However the trivial bundle case is easy by the Kunneth formula of homology. □

Before we compute the homotopy groups of $MO$, we first compute its mod 2 homology.

**Theorem 4.1.13.** $H_*(MO) = \mathbb{Z}/2\mathbb{Z}[a_1, a_2, \dots]$.

*Proof.* By theorem 4.1.12 above, we have a stablized version $H_*(MO) \simeq H_*(BO)$. Moreover, by construction, we find that the Thom isomorphism are compatible with the ring spectra structure, hence it is an isomorphism of graded rings. Then it suffices to compute $H_*(BO)$.

Notice the multiplication of $H_*(BO)$ induces the comultiplication on $H^*(BO) \simeq \mathbb{Z}/2\mathbb{Z}[w_1, \dots]$ given by

$$w_i \mapsto \sum_{j+k=i} w_j \otimes w_k$$

Then $H_*(BO)$ is thus its dual, and the computations gives $H_*(BO) = \mathbb{Z}/2\mathbb{Z}[b_i]$, where $b_i$ is the image of $w_1^i \in H_i(BO(1))$ under the inclusion $H_i(BO(1)) \to H_i(BO)$. Thus if we let $a_i$ be the preimage of $b_i$ under the isomorphism, we obtain the results. □

*Remark* 11. Since $b_i$ comes from $w_1^i$, and there is a slant product involved in the Thom isomorphism, we have $a_i$ coming from $w_1^{i+1}$, under the natural map $H_*(MO(1)) \to H_*(MO)$ and the homotopy equivalence $BO(1) \simeq MO(1)$.

Now, since the cohomology of is an $\mathcal{A}$ module, the homology is naturally an $\mathcal{A}_*$ comodule (notice that in mod 2 coefficients, $H_*$ is the dual of $H^*$ since the homology is finite dimensional). We will see that the calculation of this comodule structure of certain spaces finishes the computation of $\pi_*(MO)$.

**Lemma 4.1.14.** *Suppose the comodule structure of $H_*(BO(1))$ is given by $\gamma : H_*(BO(1)) \to \mathcal{A}_* \otimes H_*(BO(1))$, and $\gamma(w_1^i) = a_{i,j} \otimes w_1^j$. Then we have*

$$a_{i,1} = \begin{cases} \xi_r & \text{if } i = 2^r \\ 0 & \text{otherwise} \end{cases}$$

*and $a_{i,i} = 1$.*

*Proof.* By direct computation. □

**Theorem 4.1.15.** *We let $N_* = \mathbb{Z}/2\mathbb{Z}[u_i]$ for all $i > 1$ and $i \neq 2^i - 1$, with the degree of $u_i$ being $i$. Then we define a homomorphism $f : H_*(MO) \to N_*$ by*

$$f(a_i) = \begin{cases} 0 & \text{if } i = 2^r - 1 \\ u_i & \text{otherwise} \end{cases}$$

*Then the composite $g = (1 \otimes f) \circ \gamma : H_*(MO) \to \mathcal{A}_* \otimes H_*(MO) \to \mathcal{A}_* \otimes N_*$ is an isomorphism of $\mathcal{A}$-comodules and $\mathbb{Z}/2\mathbb{Z}$-algebras.*

*Proof.* $g$ is clearly a homomorphism of $\mathcal{A}$-comodules and $\mathbb{Z}/2\mathbb{Z}$-algebras. We prove it is an isomorphism. Notice that both the source and target of $g$ are polynomial algebras with one generator of degree $i$ for all $i$, by the structure of $\mathcal{A}_*$ stated in theorem 4.1.11, hence it suffices to prove that $g$ sends generators to generators. Since $a_i = \iota(w_1^i)$, we may use this to compute $\gamma(a_i) = \iota(\gamma(w_1^{i+1}))$. Then, let $I$ be the ideal of $\mathcal{A}_* \otimes H_*(MO)$ generated by elements that are products of at least two generators. Then we have by lemma 4.1.14

$$\gamma(a_i) = \iota(\gamma(w_1^{i+1})) \equiv \begin{cases} \xi_r \otimes 1 + 1 \otimes a_i & \text{if } i = 2^r - 1 \\ 1 \otimes a_i & \text{otherwise} \end{cases} \pmod I$$

This proves that the generators are sent to generators, since $\xi_r \otimes 1 + 1 \otimes a_i$ is send to $\xi_r \otimes 1$ after applying $1 \otimes f$. □

**Construction 4.1.16.** Notice that the cohomology of spheres are concentrated in two degrees, hence the Steenrod operations on spheres are trivial. Therefore, when we consider the Hurewicz homomorphism $h_{MO} : \pi_*(MO) \to H_*(MO)$, we have $\gamma \circ h_{MO}(x) = 1 \otimes h_{MO}(x)$. Thus we consider the inclusion $N_* \simeq \mathbb{Z}/2\mathbb{Z} \otimes N* \hookrightarrow \mathcal{A}_* \otimes N_*$.

Finally we are ready to prove the theorem.

**Theorem 4.1.17.** *The map $h_{MO} : \pi_*(MO) \to H_*(MO)$ is injective. Moreover, we have $g \circ h_{MO}$ an isomorphism onto $N_*$, as shown in the construction.*

*Proof.* Since each $\alpha \in H^n(MO)$ is represented by a map $MO \to \Sigma^n H\mathbb{Z}/2\mathbb{Z}$, we consider the space $\bigvee \Sigma^i H\mathbb{Z}/2\mathbb{Z}$, with each component corresponding to a generator of the cohomology of $MO$. Thus we have a natural map $f : MO \to \bigvee \Sigma^i H\mathbb{Z}/2\mathbb{Z}$. By definition, this map induces an isomorphism on mod 2 cohomology hence on homology. Then, consider the sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2^{k+1}\mathbb{Z} \longrightarrow \mathbb{Z}/2^k\mathbb{Z} \longrightarrow 0$$

and by the five lemma we have the map induce an isomorphism on $\mathbb{Z}/2^k\mathbb{Z}$ homology, and hence on the direct limit, $\mathbb{Z}[1/2]$ homology. Then, notice $\pi_*(MO)$ is of 2 torsion, hence $H_*(MO;\mathbb{Q})$ and $H_*(MO;\mathbb{Z}[1/p])$ for $p$ an odd prime are trivial. Thus by the sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \bigoplus_p \mathbb{Z}[1/p] \longrightarrow 0$$

we apply the five lemma again and find that $f$ induces an isomorphism on $\mathbb{Z}$ homology. Thus by the spectrum version of the Hurewicz theorem that $f$ must be a weak homotopy equivalence. Therefore, $f$ induces an isomorphism between $\pi_*(MO)$ and $\pi_*(\bigvee \Sigma^i H\mathbb{Z}/2\mathbb{Z})$. However by definition, $\pi_*(\bigvee \Sigma^i H\mathbb{Z}/2\mathbb{Z}) \simeq N_*$. Therefore $\pi_*(MO) = N_*$, and $h_{MO} \simeq h_{\bigvee \Sigma^i H\mathbb{Z}/2\mathbb{Z}}$ is injective. Finally the following sequence of monomorphisms gives an injection from $\pi_*(MO)$ to $N_* \subset \mathcal{A}_* \otimes N_*$, and hence a isomorphism since on each degree they have the same finite dimension.

$$\pi_*(MO) \to H_*(MO) \to \mathcal{A}_* \otimes H_*(MO) \to \mathcal{A}_* \otimes N_*$$

□

**Theorem 4.1.18.** *Two manifolds are cobordant if and only if all their Stiefel-Whitney numbers are the same. In particular, a manifold is null-bordant if and only if its Stiefel-Whitney numbers vanish.*

*Proof.* Now, since the Hurewicz map $h_{MO} : \pi_*(MO) \to H_*(MO)$ is injective, it suffices to prove that for a manifold $M$, □

## 4.2 The Hirzebruch signature theorem

**Definition 4.2.1.** Let $\Lambda$ be a ring. Then we let $A^* = \bigoplus A^n$ denote a graded $\Lambda$-algebra. We define $A^\Pi$ to be the commutative ring of all formal sums $a_0 + a_1 + \cdots$. Then, we denote the multiplicative group of formal sums to be $A^u$.
Then, Let $K_n(x_1, \dots, x_n)$ denote a sequence of polynomials with coefficients in $\Lambda$ such that $K_n(x_1, \dots, x_n)$ is homogeneous of degree $n$ if we assign $x_i$ to have degree $i$. We now define for $a \in A^u$, $K(a) \in A^u$ to be $1 + K_1(a_1) + K_2(a_1, a_2) + \cdots$. Such $K$ is called a multiplicative sequence if we have $K(a)K(b) = K(ab)$.

**Lemma 4.2.2.** *Given a formal series $f(t) = 1 + \lambda_1 t + \lambda_2 t^2 + \cdots$, there exists a unique multiplicative sequence $K$ such that $K(1+t) = f(t)$ (here $K(1+t) = 1 + K_1(t) + K_2(t, 0) + \cdots$).*

*Proof.* We first consider uniqueness. Consider the polynomial ring $\Lambda[t_1, \dots, t_n] = A^*$ with the natural grading. Then let $\sigma = (1+t_1)\cdots(1+t_n) \in A^u$. Thus $K(\sigma) = K(1+t_1)\cdots K(1+t_n) = f(t_1)\cdots f(t_n)$. Then taking the homogeneous part of degree $n$, we see that $K_n(\sigma_1, \dots, \sigma_n)$ is completely determined by the power series $f(t)$. Since $\sigma_i$ are algebraically independent, $K_n$ is unique.

Then we prove existence. We let $K_n(\sigma_1, \dots, \sigma_n) = \sum \lambda_I s_I(\sigma_1, \dots, \sigma_n)$, the right hand side to be summed over all partitions of $n$ (here $\lambda_i = \lambda_{i_1} \cdots \lambda_{i_n}$). As in lemma 3.6.9, we have

$$s_I(ab) = \sum_{JK=I} s_J(a) s_K(b)$$

Thus we have

$$\begin{aligned}
K(ab) =& \sum \lambda_I s_I(ab) \\
=& \sum_I \lambda_I \sum_{JK=I} s_J(a) s_K(b) \\
=& \sum_{JK=I} \lambda_J s_J(a) \lambda_K s_K(b) \\
=& K(a)K(b)
\end{aligned}$$

□

**Definition 4.2.3.** For $K$ a multiplicative sequence of rational coefficients and a manifold $M$ of dimension $4n$, we define its $K$-genus to be the rational number

$$K[M] = \langle K_n(p_1, \dots, p_n), \mu_M \rangle$$

Notice we have $K[M]$ a rational linear combination of the Pontrajagin numbers of $M$.
For $M$ of dimension $m \neq 4n$, we define its $K$-genus to be 0.

**Definition 4.2.4.** For a compact oriented manifold $M$ of dimension $4n$, we define its signature to be the signature of the symmetric matrix given by $H^{2n}(M;\mathbb{Q}) \times H^{2n}(M;\mathbb{Q}) \to \mathbb{Q}$, $(a,b) \mapsto \langle a \smile b, \mu_M \rangle$.

**Theorem 4.2.5.** *Let $L_k$ be the multiplicative sequence of polynomials belonging to the power series*

$$\frac{\sqrt{t}}{\tanh \sqrt{t}} = 1 + \frac{1}{3}t - \frac{1}{45}t^2 + \cdots + \frac{(-1)^{k-1} 2^{2k} B_k}{(2k)!} t^k + \cdots$$

*where $B_k$ are the Bernoulli numbers. Then we have $\sigma(M) = L[M]$ for any compact oriented manifold $M$ of dimension $4n$.*

*Proof.* □

# Part II

# Basic Stable Homotopy Theory

# Chapter 5

# Spectra

## 5.1 Basic concepts

## 5.2 The stable $\infty$-category Sp

# Chapter 6

# Constructions

## 6.1 Bousfield localization

In this section, we will develop the theory of localizations on the $\infty$-category of spectra. We begin by presenting the most important theorem of this section, the adjoint functor theorem. But before this, we need some preliminaries on $\infty$-category theory.

**Definition 6.1.1.** For an $\infty$-category $\mathcal{C}$ and a regular cardinal $\kappa$, we define $\mathcal{C}$ to be $\kappa$-filtered if for every $\kappa$-small simplicial set $K$ and every map $f : K \to \mathcal{C}$, there exists a map $\overline{f}^{\triangleright} : K^{\triangleright} \to \mathcal{C}$ extending $f$. In particular, we say $\mathcal{C}$ is filtered if $\mathcal{C}$ is $\omega$-filtered. Clearly for $\kappa < \lambda$, all $\lambda$-filtered categories are $\kappa$-filtered.
For an $\infty$-category $\mathcal{C}$ and a regular cardinal $\kappa$, we let $\mathrm{Ind}_\kappa(\mathcal{C})$ denote the full subcategory of $\mathcal{P}(\mathcal{C})$ spanned by the functors $f : \mathcal{C}^{\mathrm{op}} \to \infty\mathbf{Grpd}$ such that they classify right fibrations $\widetilde{\mathcal{C}} \to \mathcal{C}$ and $\widetilde{\mathcal{C}}$ is $\kappa$-filtered. Notice that under the intuition that $\mathcal{P}(\mathcal{C})$ is the category obtained from $\mathcal{C}$ by freely adjoining colimits, $\mathrm{Ind}_\kappa(\mathcal{C})$ has the intuition of adjoining only $\kappa$-filtered colimits. Consequently, $\mathrm{Ind}_\kappa(\mathcal{C})$ admits all $\kappa$-filtered colimits.
We define a category $\mathcal{C}$ to be accessible if it is of the form $\mathrm{Ind}_\kappa(\mathcal{C}^0)$ for some small category $\mathcal{C}^0$ and some regular cardinal $\kappa$. This definition admits a relative version: we define a functor $f : \mathcal{C} \to \mathcal{D}$ with $\mathcal{C}$ an accessible category to be accessible if it is $\kappa$-continuous for some regular cardinal $\kappa$. We define $\mathcal{C}$ to be presentable if it is both accessible and closed under small colimits.

**Theorem 6.1.2.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between presentable $\infty$-categories. Then, we have*

1. *$F$ has a right adjoint iff it preserves small colimits.*
2. *$F$ has a left adjoint iff it is accessible and preserves small limits.*

**Theorem 6.1.3.** *A stable $\infty$-category $\mathcal{C}$ is presentable iff the following conditions are satisfied.*

1. *The $\infty$-category $\mathcal{C}$ admits small coproducts.*
2. *The homotopy category $h\mathcal{C}$ is locally small.*
3. *There exists a regular cardinal $\kappa$ and a $\kappa$-compact generator $X \in \mathcal{C}$.*

*Here $X$ is a generator iff for all $Y$, $\pi_0 \mathrm{Map}_{\mathcal{C}}(X, Y) = *$ implies that $Y$ is the zero object.*

**Corollary 6.1.4.** *The category **Sp** is presentable. Moreover, for a full subcategory $\mathcal{C}$ of **Sp** closed under shifts and homotopy colimits with a small $\mathcal{C}_0$ generating $\mathcal{C}$ under colimits, $\mathcal{C}$ is also presentable.*

**Corollary 6.1.5.** *For such subcategory $\mathcal{C}$, there exists a colocalization functor $G : \mathbf{Sp} \to \mathcal{C}$.*

**Definition 6.1.6.** For $\mathcal{C}$ satisfying corollary 6.1.4, we define an object $X$ to be $\mathcal{C}$-local if for any object $Y$ in $\mathcal{C}$, we have $[Y, X] = *$. Then we define $\mathcal{C}^{\perp}$ to be the full subcategory of $\mathbf{Sp}$ spanned by $\mathcal{C}$-local objects. Finally, we define a morphism $f : X \to Y$ is a $\mathcal{C}$-equivalence if its fiber is in $\mathcal{C}$. For a ring spectrum $E$, we define $\mathcal{C}_E$ to be the subcategory spanned by the $E$-acyclic spectra (that is, $E \wedge X = *$), and we define an object to be $E$-local if it is in $\mathcal{C}_E^{\perp}$, and finally a morphism to be an $E$-equivalence if $f$ is a $\mathcal{C}_E$ equivalence.

**Construction 6.1.7.** We form the cofiber of the counit map $G(X) \to X$, denoted by $L(X)$. $L(X)$ is equipped with a map $X \to L(X)$. It is obvious that $L$ has (and is characterized by the following two properties).

1. $L(X)$ is $\mathcal{C}$-local.

2. $X \to L(X)$ is an $\mathcal{C}$-equivalence.

Moreover, $L$ is a localization into $\mathcal{C}^{\perp}$.

**Corollary 6.1.8.** *The subcategory $\mathcal{C}_E$ spanned by $E$-acyclic spectra satisfies the condition of corollary 6.1.4, hence there is a functor $G_E : \mathbf{Sp} \to \mathcal{C}_E$ exhibiting $\mathcal{C}_E$ a coreflective subcategory of $\mathbf{Sp}$. We also have $L_E$, the localization functor.*

We present a few properties of $E$-local spectra.

**Proposition 6.1.9.** *If $X$ and $Y$ are $E$-local spectra and $f : X \to Y$ is an $E$-equivalence, then $f$ is an equivalence.*

*Proof.* We first notice that if a spectrum $X$ is both $E$-local and $E$-acyclic, then $X \simeq *$. This is since $\mathrm{id} : X \to X$ is null-homotopic. Then, we may consider $\mathrm{fib}(f)$, which is both $E$-local and $E$-acyclic. □

**Proposition 6.1.10.** *$E$-local spectra are closed under retracts, shifting, forming fibers, and products.*

*Proof.* Direct. □

**Proposition 6.1.11.** *A module spectrum $X$ over a ring spectrum $E$ is $E$-local.*

*Proof.* This is since $X$ is a retract of $X \wedge E$, and $X \wedge E$ is obviously $E$-local. □

**Proposition 6.1.12.** *$E$-localization is compatible with the wedge sum, shiftings, and fiber sequences. That is, $L_E(X \vee Y) \simeq L_E(X) \vee L_E(Y)$, and $L_E(\Sigma^{\pm 1} X) \simeq \Sigma^{\pm 1} L_E(X)$.*

*Proof.* Since $E$-local spectra has products (hence finite coproducts), $L_E$ preserves finite coproducts (i.e. wedge sums). Since fiber sequences is equivalent to cofiber sequences, which is computed by pushouts, and $E$-local spectra is closed under the formation of cofibers, $L_E$ preserves fiber sequences. Then shifting is just a simple case of a fiber sequence. □

**Lemma 6.1.13.** *Let $\mathcal{C}$ be a subcategory of $\mathbf{Sp}$ satisfying corollary 6.1.4, then the following conditions are equivalent.*

1. *$\mathcal{C}^{\perp}$ is closed under colimits.*

2. *$L$ preserves colimits.*

3. *$G$ preserves colimits.*

4. *$L$ is of the form $L(X) = K \wedge X$ for some $K$.*

*Proof.* (1) $\iff$ (2) is since left adjoints preserves colimits.
(2) $\iff$ (3) by the cofiber sequence $G \to \mathrm{id} \to L$.
(2) $\iff$ (4) by observing the action of $L$ on $S$, and then generate the whole category **Sp**. $\square$

**Corollary 6.1.14.** *In this case, we have $L(X) \simeq L(S) \wedge X$.*

*Proof.* Since $X$ is $\mathcal{C}$-local iff $L(X) \simeq L(S) \wedge X \simeq *$, hence $\mathcal{C}$ can be identified with the $L(S)$-acyclic spectra. $\square$

**Definition 6.1.15.** We define a localization functor $L$ to be smashing if it satisfies the equivalent condition in lemma 6.1.13.

**Theorem 6.1.16.** *For any spectrum $E$, $L_E$ commutes with finite colimits. Hence for finite spectra $F$, we have $L_E(X \wedge F) = L_E(X) \wedge F$.*

*Proof.* It suffices to prove that the full subcategory of $E$-local spectra admits finite colimits, since $L_E$, as a left adjoint, commutes with all existing colimits. Then, for $E$-local spectra $X$ and $Y$, $X \vee Y$ is also $E$-local since for an $E$-acyclic spectrum $Z$, a morphism $Z \to X \vee Y$ is equivalent to the data of a morphism $Z \to X$ and $Z \to Y$, hence $Z \to X \vee Y$ must be trivial. Thus $E$-local spectra admits all finite limits (stable categories admits all pushouts). As for the second assertion, to show that $L_E(X) \wedge F \to L_E(X \wedge F)$ is an equivalence, it suffices to show that if $W$ is $E$-local then $W \wedge F$ is $E$-local. Let $Z$ be an $E$-acyclic spectrum. Then, we have $\mathrm{Map}(Z, W \wedge F) \simeq \mathrm{Map}(Z \wedge DF, W) \simeq *$, since $Z \wedge DF$ is still $E$-acyclic. Thus $W \wedge F$ is $E$-local. $\square$

**Lemma 6.1.17.** *Localization functors are uniquely determined by their images. Hence all localization functors form a poset with order the inclusion of their images.*

*Proof.* Direct by verifying the universal properties. $\square$

**Proposition 6.1.18.** *If $\mathcal{C}_F \subseteq \mathcal{C}_E$, then we have the following.*

1. *Each $E$-local spectrum is $F$-local.*

2. *For each spectrum $X$ we have natural equivalences $L_E(L_F(X)) \simeq L_E(X) \simeq L_F(L_E(X))$, $G_F(L_E(X)) \simeq 0 \simeq L_E(G_F(X))$, $G_E(G_F(X)) \simeq G_F(X) \simeq G_F(G_E(X))$, and finally $G_E(L_F(X)) \simeq L_F(G_E(X))$.*

3. *For each $X$ we have fiber sequences $G_E(L_F(X)) \to L_F(X) \to L_E(X)$ and $G_F(X) \to G_E(X) \to G_E(L_F(X))$.*

*Proof.* The first two properties are direct. The last one follows from applying $L_F$ to the fiber sequences $G_E(X) \to X \to L_E(X)$ and $G_E$ to $G_F(X) \to X \to L_F(X)$, and apply the second property. $\square$

**Definition 6.1.19.** We define two spectra $E$ and $F$ to be Bousfield equivalent if their localization functors are isomorphic. The equivalence classes are called Bousfield classes. We denote the Bousfield class of $E$ by $\langle E \rangle$.

**Construction 6.1.20.** We notice that we may form an algebraic sturcture on all Bousfield classes resembling a lattice. We define for two spectra $E$ and $F$, $\langle E \rangle \leq \langle F \rangle$ iff all $F$-acyclic spectra are $E$-acyclic. Then we notice $\langle * \rangle$ is the smallest Bousfield class, while $\langle S \rangle$ is the largest (here $S$ is the sphere spectrum).
Moreover, $\langle E \rangle \vee \langle F \rangle := \langle E \vee F \rangle$ is a minimal upper bound for $\langle E \rangle$ and $\langle F \rangle$, while $\langle E \rangle \wedge \langle F \rangle := \langle E \wedge F \rangle$ is a not necessarily maximal lower bound for $\langle E \rangle$ and $\langle F \rangle$.
Finally, there exists a dual of a spectrum $E^{\perp}$ such that $\langle E \rangle \wedge \langle E^{\perp} \rangle = \langle 0 \rangle$ and $E^{\perp\perp} \simeq E$.

# Chapter 7

# Spectral sequences

## 7.1 The Atiyah-Hirzebruch spectral sequence

## 7.2 The Milnor spectral sequence

## 7.3 The Adams spectral sequence

# Chapter 8

# Theorems

## 8.1 Bott periodicity

In this section, we prove the famous result stated first by Bott. We will use a computational proof, so that we can get more information on the homology of relavent spaces. There are, indeed, proofs that does not involves too much computation, such as the proof using group completion.

**Theorem 8.1.1.** *We have a map $\beta : BU \to \Sigma SU$ inducing a homotopy equivalence.*

Before we do anything, we state the comparison theorem, which will be extremely useful in the proof.

**Theorem 8.1.2.** *Let $f : E \to' E$ be a homomorphism of multiplicative first quadrant homological spectral sequences. Assume that we have the following commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & E^2_{p,0} \otimes E^2_{0,q} \arrow[r] \arrow[d] & E^2_{p,q} \arrow[r] \arrow[d] & \mathrm{Tor}_1(E^2_{p-1,0}, E^2_{0,q}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & {}'E^2_{p,0} \otimes {}'E^2_{0,q} \arrow[r] & {}'E^2_{p,q} \arrow[r] & \mathrm{Tor}_1({}'E^2_{p-1,0}, {}'E^2_{0,q}) \arrow[r] & 0 \end{tikzcd}$$

*Then we have two of the following statements imply the third.*

1. *$f^2_{p,0} : E^2_{p,0} \to {}'E^2_{p,0}$ is an isomorphism for all $p \geq 0$.*
2. *$f^2_{0,q} : E^2_{0,q} \to {}'E^2_{0,q}$ is an isomorphism for all $q \geq 0$.*
3. *$f^\infty_{p,q} : E^\infty_{p,q} \to {}'E^\infty_{p,q}$ is an isomorphism for all $p, q \geq 0$.*

We first construct the map $\beta$.

**Construction 8.1.3.** We let $\mathbb{V}$ denote $\mathbb{C}^\infty$, and $U$ denote the colimit of $U(n)$ under the standard inclusion $\mathbb{C}^n \hookrightarrow \mathbb{C}^\infty$, and $SU(n)$ denote the colimit of $SU(n)$ (regarded as the standard subgroup $SU(n) \hookrightarrow U(n)$). Finally, we let $BU = U/U \times U$ denote the colimit of the Grassmannians $U(2n)/U(n) \times U(n)$. Also, we consider the loops of length $\pi$ for convenience.
For $0 \leq \theta \leq \pi$, we define $\nu(\theta) \in U(\mathbb{V}^2)$ by $\nu(\theta)(z, z') = (e^{i\theta} z, e^{-i\theta} z')$. Then, we define $\beta : U(\mathbb{V}^2) \to \Omega SU(\mathbb{V}^2)$ by

$$\beta(T)(\theta) = [T, \nu(\theta)]$$

where the $[-,-]$ denote the commutator. This is clearly a loop. Moreover, since $\nu(\theta)$ is scalar multiplication on each component, we have when $T \in U(\mathbb{V}) \times U(\mathbb{V})$ the loop is trivial. Hence by passing to quotients, we have defined a map $\beta : BU = U/U \times U \to \Omega SU$. Clearly, $\beta$ is a map of H-spaces.

**Proposition 8.1.4.** *We have $H_*(BU;\mathbb{Z}) = \mathbb{Z}[\gamma_1, \gamma_2, \dots]$, where $\gamma_i \in H_i(BU(1))$ is dual to $c_1^i$.*

*Proof.* This proof is exactly the same as in the computation of $H_*(BO;\mathbb{Z}/2\mathbb{Z})$. Essentially, the hopf algebra structure of the cohomology of these spaces are clear, and it suffices to calculate their dual, resulting in the conclusion. □

**Proposition 8.1.5.** $H_*(SU(n);\mathbb{Z}) \simeq \Lambda_{\mathbb{Z}}[x_3, x_5, \dots, x_{2n-1}]$ *(here the exterior product is viewed as a Hopf algebra).*

*Proof.* We first calculate $H^*(SU(n);\mathbb{Z})$. We do this inductively. The case for $n = 1$ and $2$ is trivial. Then we consider the fibration $SU(n) \to SU(n+1) \to S^{2n+1}$. We apply the Leray-Serre spectral sequence on this fibration. The second page of this spectral sequence has two rows, one at $0$ and one at $2n+1$, with each row the homology of $SU(n)$ at the corresponding position. Then, the first non-trivial differential that may exists is from $(0, 2n) \to (2n+1, 0)$. However, this spectral sequence is multiplicative, and by the induction hypothesis, any element in $(2n+1, 0)$ must comes from a product of elements in $(m_i, 0)$, which, by the structure of the spectral sequence, are permanent cycles. Therefore the spectral sequence degenerates, and we obtain $H^*(SU(n);\mathbb{Z}) \simeq \Lambda_{\mathbb{Z}}[\alpha_3, \alpha_5, \dots, \alpha_{2n-1}]$.
Then we must compute the Hopf algebra structure of $H^*(SU(n);\mathbb{Z})$, and dualize it to obtain our conclusion. In fact, it suffices to prove each $\alpha_i$ is primitive, which then gives the result directly by setting $x_i$ to be the dual of $\alpha_i$. We again use the induction hypothesis. To prove the case for $SU(n+1)$, we first see that $\alpha_3, \dots, \alpha_{2n-1}$ are primitive by the induction hypothesis. Then, notice that $\alpha_{2n+1}$ is the image of the generator for $\pi_{2n+1}(S^{2n+1})$. Then since $SU(n) \to SU(n+1) \to S^{2n+1}$ is a fibration, we see that this element is also primitive (there is no room for the coproduct to be too large with the assumption that $H^*(SU(n+1)) \to H^*(SU(n))$ is a Hopf algebra map). □

**Proposition 8.1.6.** $H_*(\Omega SU(n);\mathbb{Z}) \simeq \mathbb{Z}[\delta_1, \dots, \delta_n]$, *where $\delta_i$ has degree $2i$. Note that here the polynomial is also viewed as a Hopf algebra.*

*Proof.* The main idea is to apply the Leray-Serre spectral sequence on the path space fibration $\Omega SU(n) \to PSU(n) \to SU(n)$, say the spectral sequence is $F$. Then, we construct a spectral sequence

$$E^2_{*,*} = \mathbb{Z}[\delta_1, \dots, \delta_n] \otimes \Lambda_{\mathbb{Z}}[x_3, \dots, x_{2n+1}]$$

with the $\delta_i$'s concentrated on the bottom with degree $(0, 2i)$ and $x_{2k+1}$'s concentrated on the left with degree $(2k+1, 0)$. The other terms are given by obvious tensors. Then we let the cycles $x_{2i+1}$ being mapped to $\delta_i$ at the $2i$-th page, which would ensure that this spectral sequence degenerates with only $E^\infty_{0,0} = \mathbb{Z}$. Then we have an obvious map from $E$ to $F$, which is the identity on $E^2_{p,0}$. This, along with the comparison theorem (theorem 8.1.2), gives the calculation $H_*(\Omega SU(n);\mathbb{Z}) \simeq \mathbb{Z}[\delta_1, \dots, \delta_n]$. □

*Proof.* Now that we have seen the homology of $\Omega SU$ and $BU$ are both polynomial algebras on infinitely many generators, with one on degree $2i$, we show that $\beta$ induces an isomophism on homology. In fact, since we have an obvious map $\mathbb{CP}^\infty \to BU \to \Omega SU$, it suffices to show that $c_1^i$ is mapped to $\delta_i$ modulo decomposables, since the image of $c_1^i$ in $BU$ is a set of generators. We first notice that $\beta$ induces a morphism of fibrations by restricting the map $\beta$ to $\mathbb{CP}^n \hookrightarrow BU(1)$. Explicitly, we have

$$\begin{tikzcd}
\mathbb{CP}^{n-1} \arrow[r, "i"] \arrow[d, "\alpha"'] & \mathbb{CP}^n \arrow[r, "\rho"] \arrow[d, "\alpha"'] & S^{2n} \arrow[d, "h"] \\
\Omega SU(n) \arrow[r, "\Omega i"'] & \Omega SU(n+1) \arrow[r, "\Omega \pi"'] & \Omega S^{2n+1}
\end{tikzcd}$$

here the adjoint map of $h$ is the isomorphism $\Sigma S^{2n} \simeq S^{2n+1}$.
To see that $c_1^i$ is mapped to $\delta_i$ modulo decomposables, we consider the following diagram.

$$\begin{tikzcd}
H_{2n}(\mathbb{CP}^n) \arrow[rrr, "\alpha_*"] \arrow[ddddd, "\simeq"'] \arrow[ddr, "\simeq"] & & & H_{2n}(\Omega SU(n+1)) \arrow[dl, "\simeq"'] \arrow[ddl, "\sigma"] \arrow[ddddd, "(\Omega\pi)_*"] \\
& & H_{2n+1}(\Sigma\Omega SU(n+1)) \arrow[d, "\varepsilon_*"'] & \\
& H_{2n+1}(\Sigma\mathbb{CP}^n) \arrow[ur, "(\Sigma\alpha)_*"] \arrow[r, "\hat{\alpha}_*"'] \arrow[d, "\simeq"'] & H_{2n+1}(SU(n+1)) \arrow[d, "\pi_*"] & \\
& H_{2n+1}(\Sigma S^{2n}) \arrow[r, "\hat{h}_*"] \arrow[dr, "(\Sigma h)_*"'] & H_{2n+1}(S^{2n+1}) & \\
& & H_{2n+1}(\Sigma\Omega S^{2n+1}) \arrow[u, "\varepsilon_*"] & \\
H_{2n}(S^{2n}) \arrow[rrr, "h_*"'] \arrow[uur, "\simeq"] & & & H_{2n}(\Omega S^{2n+1}) \arrow[uul, "\sigma"'] \arrow[ul, "\simeq"]
\end{tikzcd}$$

Now, the generator $\delta_n$ (in the top right corner) is mapped to the fundamental class in $H_{2n+1}(S^{2n+1})$ under $\pi_*\sigma$ by definition, and so is $c_1^i \in H_{2n}(\mathbb{CP}^\infty)$ (in the top left corner). But the maps connecting the top left corner and $H_{2n+1}(S^{2n+1})$ is an isomorphism, hence we obtain the results. $\square$

## 8.2 The nilpotence theorem

**Theorem 8.2.1.** *Let $E$ be a ring spectrum, then we have the kernel of the map $h : \pi_*(E) \to MU_*(E)$ consisting of nilpotent elements.*

**Corollary 8.2.2.** *Every element of positive degree in $\pi_*(S)$ is nilpotent.*

*Proof.* The proof of the theorem will have the following outline. We will first prove the case for $E$ a connective ring spectrum of finite type. To do this, we construct a sequence of spectra $X(n)$ with $X(1) = S$ and $X(\infty) = MU$, and $X(\infty) = \varinjlim X(n)$. Then, let $\alpha$ be an element in the kernel of $h$ of degree $d$. By definition $\alpha \in MU_*(X) = 0$. $\square$

### 8.2.1 Preliminary results

**Lemma 8.2.3.** *For a ring spectrum $X$ and $x \in \pi_d(X)$, we let $x^{-1}X$ denote the colimit $\varinjlim(X \xrightarrow{x} \Sigma^{-d}X \xrightarrow{x} \Sigma^{-2d}X \to \cdots)$. Then for any ring spectrum $E$, we have $x^{-1}X$ is $E$-acyclic is equivalent to the image of $x$ is nilpotent in $E_d(X)$.*

*Proof.* $\Longrightarrow$ : Since $E \wedge x^{-1}X = 0$, and the smash product commutes with colimits, we have $\varinjlim(E \wedge X \to \Sigma^{-d}E \wedge X \to \cdots) = 0$. Then since $S$ is compact, $[S, -]$ commutes with filtered colimits, hence $\varinjlim(E_*(X) \to E_{*+d}(X) \to \cdots) = 0$. This, however, is exactly the localization of $E_*(X)$ at the image of $x$. Hence the vanishing of the localization implies its nilpotency.
$\Longleftarrow$ : Essentially by reversing the above proof. $\square$

### 8.2.2 The construction of $X(n)$ and $G_j$

**Construction 8.2.4.** We define $X(n)$ to be the Thom spectrum associated to the map $\Omega SU(n) \to \Omega SU \simeq BU$, where the second map is since we have a fibration $SU(n) \to U(n) \to S^1$

inducing isomorphisms on homotopy groups of degree $> 1$, hence by Bott periodicity, the second map is a homotopy equivalence. In this sense, $MU$ is the Thom spectra associated to the map $BU \to BU$, which can be regarded as the map $\Omega SU \to BU$. Clearly we have $\varinjlim X(n) \simeq MU$.

**Lemma 8.2.5.** *$X(n)$ and $MU$ are commutative ring spectra with $H_*(X(n);\mathbb{Z}) \simeq \mathbb{Z}[t_1, \dots, t_{n-1}]$ and $H_*(MU;\mathbb{Z}) \simeq \mathbb{Z}[t_1, t_2, \dots]$.*

*Proof.* Recall that from the stable Thom isomorphism, we have $H_*(X(n);\mathbb{Z}) \simeq H_*(\Omega SU(n);\mathbb{Z})$, and from the calculation proposition 8.1.6, we obtain the results immediately. Then $H_*(MU;\mathbb{Z}) \simeq H_*(\Omega SU;\mathbb{Z}) = \mathbb{Z}[t_1, t_2, \dots]$. $\square$

**Corollary 8.2.6.** *The map $X(n) \to MU$ is $(2n-1)$-connected.*

*Proof.* The map induces an isomorphism of homology groups until the $(2n-1)$-th degree. $\square$

**Definition 8.2.7.** For a based space $(X, *)$, we recall that the James construction of $X$ is defined as

$$JX = \coprod_{j=0}^{\infty} X^j / \sim$$

where $\sim$ is the equivalence relation $(x_1, \dots, x_{i-1}, *, x_{i+1}, \dots, x_j) \sim (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_j)$. Or equivalently, this is the (non-commutative) free topological monoid generated by $X$. We define $J_kX$ to be the $k$-th stage of this construction, that is,

$$J_kX = \coprod_{j=0}^{k} X^j / \sim$$

Fundamental results on the James construction is the Dold-Thom theorem and the stable splitting.

**Theorem 8.2.8.** *We have a natural weak homotopy equivalence $\lambda : J(X) \to \Omega\Sigma X$ for $X$ a CW complex.*

**Theorem 8.2.9.** *For any connected CW complex $X$, we have*

$$\Sigma J_kX = \bigvee_{j=0}^{k} \Sigma X^{\wedge j}, \qquad \Sigma JX = \bigvee_{j=0}^{\infty} \Sigma X^{\wedge j}$$

**Construction 8.2.10.** Recall that we have a fibration $\Omega SU(n) \to \Omega SU(n+1) \to \Omega S^{2n+1}$. Then notice by theorem 8.2.8, we have $\Omega S^{2n+1} \simeq JS^{2n}$. We define $B_k$ to be the pullback $\Omega SU(n)$-bundle

$$\begin{tikzcd} B_k \arrow[r, "i_{-1,0}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega SU(n+1) \arrow[d] \\ J_kS^{2n} \arrow[r] & JS^{2n} \end{tikzcd}$$

Then, we let $F_k$ denote the Thom spectrum associated to the map $B_k \to \Omega SU(n+1) \to \Omega SU \simeq BU$. Finally we let $G_j$ (for the fixed prime $p$) to be $(F_{p^j-1})_{(p)}$.

**Lemma 8.2.11.** *Notice the filtration $J_0S^{2n} \to J_1S^{2n} \to \cdots \to JS^{2n}$ induces maps*

$$\Omega SU(n) \simeq B_0 \to B_1 \to \cdots \to \Omega SU(n+1)$$

*Passing to Thom spectra, we have maps*

$$X(n) \simeq F_0 \to F_1 \to \cdots \to X(n+1)$$

*We claim that the spectra $F_k$ are $X(n)$-module spectra whose homology is a free $H_*(X(n);\mathbb{Z})$-module, regarded as a submodule of $H_*(X(n+1),\mathbb{Z}) \simeq \mathbb{Z}[x_1, \dots, x_n]$ generated by $1, x_n, \dots, x_n^k$.*

*Proof.* We prove only the result in the topological space case, and use the Thom isomorphism to pass to the Thom spectra. Notice we have a fiber sequence

$$\Omega SU(n) \to B_k \to J_k S^{2n} \to SU(n)$$

Then we replace the last map $J_k S^{2n} \to SU(n)$ by a fibration $E \to SU(n)$

$$\begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & PSU(n) \arrow[d] \\ J_k S^{2n} \arrow[r] & SU(n) \end{tikzcd}$$

Then the loopspace $\Omega SU(n)$ acts naturally on the fiber of this fibration $E \to SU(n)$, namely, $B_k$. This, after passing to Thom spectra, gives a natural action of $X(n)$ on $F_k$, which endows it with a $X(n)$-module structure. Then we consider the Leray-Serre spectral sequence associated to the fiber sequence $\Omega SU(n) \to B_k \to J_k S^{2n}$. By theorem 8.2.9, we have

$$\begin{aligned} &\widetilde{H}_*(J_k S^{2n}) \\ \simeq &\widetilde{H}_{*+1}(\Sigma J_k S^{2n}) \\ \simeq &\widetilde{H}_{*+1}(\bigvee_{j=0}^{k} \Sigma S^{2nj}) \\ \simeq &\bigoplus_{j=0}^{k} \widetilde{H}_*(S^{2nj}) \end{aligned}$$

Thus we have $H_*(J_k S^{2n})$ is free of two generators of dimension 0 and one generators of dimension $2i$ for $1 \le i \le k$. The fiber has homology $\mathbb{Z}[\delta_1, \ldots, \delta_n]$ by proposition 8.1.6. Thus the Leray-Serre spectral sequence degenerates (since all terms are of even degrees), and gives the homology of $B_k$ as desired. $\square$

# Part III

# Chromatic Homotopy Theory

# Chapter 9

# The Lazard Ring and Quillen's Theorem

## 9.1 Formal group laws and the Lazard ring

**Definition 9.1.1.** For a commutative ring $R$, we define a formal group law over $R$ to be a formal power series $f \in R[[x, y]]$, satisfying the following identities.

1. (Unit) $f(x, 0) = f(0, x) = x$.
2. (Commutativity) $f(x, y) = f(y, x)$.
3. (Associativity) $f(x, f(y, z)) = f(f(x, y), z)$.

We denote $\mathrm{FGL}(R)$ by the set of all formal group laws over $R$. For a morphism of rings $\varphi : R \to R'$ and a formal group law $f$ over $R$, we denote $\varphi_* f$ as the formal group law over $R'$ givne by the map $\varphi_* : R[[x, y]] \to R'[[x, y]]$. As a consequence, FGL is a functor from **Ring** to **Set**.

**Proposition 9.1.2.** *The functor* FGL *is corepresentable. That is, there is a ring $R$ and a formal group law $f_{univ}$ over $R$ such that for any ring $S$ and a formal group law $g$ over $S$, there is a unique map $\varphi : R \to S$ such that $\varphi_* f_{univ} = g$.*

*Proof.* Notice that for $f \in R[[x, y]] = \sum a_{i,j} x^i y^j$ to be a formal group law, the three conditions translates to be $a_{i,0} = a_{0,j} = 0$ for $i, j \geq 1$, $a_{i,j} = a_{j,i}$, and the associativity to a complicated relation we won't write down here. Therefore, we may simply let $R = \mathbb{Z}[c_{i,j}]/I$, where $I$ is the ideal generated by the above relations. □

**Definition 9.1.3.** The universal ring is denoted by $L$, the lazard ring, and the formal group law $f_{univ}$.

Our goal of this section is to characterize the structure of the ring $L$. In fact, we have the following theorem.

**Theorem 9.1.4.** *$L \simeq \mathbb{Z}[t_1, t_2, \ldots]$, where $t_i$ has degree $2i$.*

The first thing is to explain where the grading comes from.

*Remark* 12. We let the degree of $c_{i,j}$ to be $2(i + j - 1)$. Therefore, if we let $x$ and $y$ have degree $-2$, then $\sum c_{i,j} x^i y^j$ has degree $-2$. This grading comes from topology, where $x$ and $y$ are often the first Chern classes, and $f(x, y)$ is often the first Chern class of the tensor product bundle. Actually, we may consider the multiplicative group scheme $\mathbb{G}_m = \operatorname{Spec} R[t, t^{-1}]$. Notice that we have an action of $R^\times$ on $\mathrm{FGL}(R)$, given by $f(x, y) \mapsto \lambda^{-1} f(\lambda x, \lambda y)$. Later, for a graded ring $R$, we will denote the set of formal group laws on $R$ respecting the gradings by $\mathrm{FGL}^{\mathrm{gr}}(R)$. Clearly, this is corepresented by the set of graded maps $\mathrm{Hom}^{\mathrm{gr}}(L, R)$.

**Construction 9.1.5.** We first construct a map $\varphi : L \to \mathbb{Z}[b_1, b_2, \dots]$. We consider the formal power series $g(x) = x + b_1x^2 + b_2x^3 + \cdots$. Then we have a formal group law over $\mathbb{Z}[b_1, b_2, \dots]$ given by $f(x, y) = g(g^{-1}(x) + g^{-1}(y))$. Notice if we let $x, y$ have degree -2, then we result in an element of degree 2. We define $\varphi$ as this classifying map, which is a map of graded rings.

The classifying map $\varphi$ turns out to have a very interesting property: it is an isomorphism with rational coefficients, but not with $\mathbb{F}_p$ coefficients.

**Lemma 9.1.6.** *Let $I$ be the ideal in $L$ generated by elements of positive degree, and $J$ the ideal in $\mathbb{Z}[b_1, b_2, \dots]$ generated by elements of positive degree. Notice $J/J^2$ is a free abelian group with one generator (the image of $b_i$) of degree $2i$. Also, notice $I$ inherits a natural grading from $L$. Then, we have $\varphi_{2n} : (I/I^2)_{2n} \to (J/J^2)_{2n} \simeq \mathbb{Z}$ injective, with image $p\mathbb{Z}$ if $n$ is of the form $p^r - 1$ and $\mathbb{Z}$ otherwise.*

We postpone the proof of this lemma and first use it to prove theorem 9.1.4.

*Proof.* We now prove theorem 9.1.4. By lemma 9.1.6, the abelian group $(I/I^2)_{2n}$ is isomorphic to $\mathbb{Z}$. Therefore we may choose $t_i \in L_{2n}$ lifting the generator of $(I/I^2)_{2n}$. Therefore we obtain a map of graded rings $\theta : \mathbb{Z}[t_1, \dots] \to L$.
We prove $\theta$ is surjective. We prove by induction that $\theta$ induces a surjection in degree $2n$. Now, the induction hypothesis shows that the image of $\theta$ contains $(I^2)_{2n}$. Also, the image of $\theta$ contains a generator for $(I/I^2)_{2n}$, hence it contains $I_{2n} = L_{2n}$.
We finally prove $\theta$ is injective. In fact, it suffices to prove the composite $\psi = \varphi \circ \theta : \mathbb{Z}[t_1, \dots] \to L \to \mathbb{Z}[b_1, \dots]$ is a rational isomorphism. In fact, since $\psi_{\mathbb{Q}}$ is a map of graded rings, and for each $n$ we have $\mathbb{Q}[t_1, \dots]_n$ and $\mathbb{Q}[b_1, \dots]_n$ a finite $\mathbb{Q}$-vector space of the same finitely many dimensions, it suffices to prove $\psi_{\mathbb{Q}}$ is surjective. This is done by letting $I'$ be the ideal of $\mathbb{Q}[t_1, \dots]$ generated by $t_i$. Again by lemma 9.1.6, we have a surjection $I'/(I')^2 \to (J/J^2) \otimes \mathbb{Q}$. By repeating the proof of the surjectivity of $\theta$, we have $\psi_{\mathbb{Q}}$ is surjective. □

Now, to prove lemma 9.1.6, we first need to identify the abelian group $(I/I^2)_{2n}$. We will determine this group by proving that for any Abelian group $M$, we have $\mathrm{Hom}((I/I^2)_{2n}, M) \simeq M$. We denote the functor corepresented by $(I/I^2)_{2n}$ by $F(M)$ for a fixed $n$. Obviously, $F$ is related to the functor FGL.

**Construction 9.1.7.** In this section, for any Abelian group $M$, we let $\mathbb{Z} \oplus M$ denote the graded ring with the 0-th grade $\mathbb{Z}$ and the $2n$-th grade $M$, with multiplication law given by $(a, m)(a', m') = (aa', am' + a'm)$. Therefore, we have $\mathrm{FGL}^{\mathrm{gr}}(\mathbb{Z} \oplus M) \simeq \mathrm{Hom}^{\mathrm{gr}}(L, \mathbb{Z} \oplus M) \simeq F(M)$. In other words, $F(M)$ can be identified with the following formal group laws

$$f(x, y) = x + y + \sum_{i+j=n+1} m_{i,j} x^i y^j$$

where $m_{i,j} \in M$. For $f(x, y)$ of this form to be a formal group law, the three requirements translates into the following.

1. $m_{n+1,0} = m_{0,n+1} = 0$.
2. $m_{i,j} = m_{j,i}$.
3. $$\binom{i+j}{j} m_{i+j,k} = \binom{j+k}{j} m_{i,j+k}$$

Therefore, we have now translated the problem into solving the following equations in an Abelian group $M$.

**Construction 9.1.8.** Recall that we have a map $(I/I^2)_{2n} \to (J/J^2)_{2n} \simeq \mathbb{Z}$. We consider the map

$$\lambda : M \simeq \mathrm{Hom}((J/J^2)_{2n}, M) \to \mathrm{Hom}((I/I^2)_{2n}, M) \simeq F(M)$$

Here, notice that $\mathrm{Hom}((J/J^2)_{2n}, M)$ is equal to $\mathrm{Hom}^{\mathrm{gr}}(\mathbb{Z}[b_1, \dots], \mathbb{Z} \oplus M)$, and $m$ corresponds to the map $\psi_m : \mathbb{Z}[b_1, \dots] \to \mathbb{Z} \oplus M$ sending $b_n$ to $m$ and other $b_i$ to 0. Therefore the change of variables corresponding to $\psi_m$ is $g(x) = x + mx^{n+1}$. Since $m^2 = 0$ in $\mathbb{Z} \oplus M$, the inverse of $g(x)$ is obviously $g^{-1}(x) = x - mx^{n+1}$. Thus $g$ defines the formal group law

$$f(x,y) = g(g^{-1}(x)+g^{-1}(y)) = g(x-mx^{n+1}+y-my^{n+1}) = x+y+m((x+y)^{n+1}-x^{n+1}-y^{n+1})$$

Therefore, $\lambda$ takes $m$ to the sequence $m_{i,j}$, where $m_{i,j} = \binom{n+1}{i}m$ for $i, j \neq 0$.
However, there can be more solutions. For example, if $\binom{n+1}{i}$ has a common divisor $d$, then $m_{i,j}/d$ is also a valid solution.

We then try to prove lemma 9.1.6. Before this, we need some results in elementary number theory.

**Lemma 9.1.9.** *Suppose $a$ and $b$ are integers with base $p$ expansions $a = \sum a_i p^i$, $b = \sum b_i p^i$. Then we have*

$$\binom{a}{b} = \prod \binom{a_i}{b_i}$$

*Proof.* This is elementary. ☐

**Corollary 9.1.10.** *$\binom{i+j}{i}$ is not divisible by $p$ iff each digit of the $p$-base expansion of $i+j$ is as large as $i$. In other words, iff $i+j$ is calculated without carrying.*

**Corollary 9.1.11.** *Let $d$ be the greatest common divisor of $\binom{n+1}{i} \neq 1$. Then $d = p$ if $n+1 = p^r$ and $1$ otherwise.*

**Proposition 9.1.12.** *Let $\lambda' : M \to F(M)$ be the map carrying $m$ to the sequence $m_{i,j} = \binom{n+1}{i}m/d$. Then $\lambda'$ is an isomorphism. That is, lemma 9.1.6 holds.*

*Proof.* It suffices to prove that $\lambda'$ is an isomorphism after localizing at every $p$. Therefore, we may assume $M$ is a $\mathbb{Z}_{(p)}$ module. ☐

## 9.2 Complex oriented cohomology theories

We begin with the definition of a complex oriented cohomology theory.

**Definition 9.2.1.** For a multiplicative cohomology theory represented by a ring spectrum $E$, we say that it is complex orientable if the map $\iota : \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$ induces an surjection $\iota^* : E^2(\mathbb{CP}^\infty) \to E^2(\mathbb{CP}^1)$.
Recall that $E^n(X) \simeq \tilde{E}^n(X) \oplus E^n(*)$. Then notice that $\tilde{E}^2(\mathbb{CP}^1) = \tilde{E}^2(S^2) = \tilde{E}^0(S^0) = E^0(*) = \pi_0(E)$ has a canonical unit. Thus equivalently, $E$ is complex orientable iff there is a $t \in \tilde{E}^2(\mathbb{CP}^\infty)$ such that $t$ restricts to the canonical unit element of $\tilde{E}^2(S^2)$. We define a complex orientation of $E$ as a choice of $t$. In this case, we say $E$ is complex oriented.

Often, for nice spaces $X$, the existence of a complex orientation of $E$ forces the Atiyah Hirzebruch spectral sequence for $E^*(X)$ to degenerate.

**Theorem 9.2.2.** *Let $X$ be a space and assume that each of the homology group $H_n(X;\mathbb{Z})$ is a free abelian group on generators $\{h_{\alpha,n}\}_{\alpha \in B_n}$. Let $c_{\alpha,n}$ be the dual basis in $H^n(X;\mathbb{Z})$. Let $E$ be a ring spectrum and let $\tau_{\leq 0}E$ denote its truncation. Then the unit map $S \to E$ induces a map $H\mathbb{Z} \simeq \tau_{\leq 0}S \to \tau_{\leq 0}E$. Then, the homology classes $h_{n,\alpha}$ have images $h'_{n,\alpha} \in (\tau_{\leq 0}E)_n(X)$ and the cohomology classes $c_{n,\alpha}$ have images $c'_{n,\alpha} \in (\tau_{\leq 0}E)^n(X)$. Assume that one of the following conditions holds:*

1. *Each $h'_{n,\alpha}$ can be lifted to a class $h''_{n,\alpha} \in E_n(X)$ under the map $\tau_{\le 0}E \to E$.*
2. *Each $c'_{n,\alpha}$ can be lifted to a class $c''_{n,\alpha} \in E_n(X)$, and $H_n(X, \mathbb{Z})$ is finitely generated.*

*Then we have the following:*

1. *The smash product $E \wedge \Sigma^\infty X_+$ is equivalent as an $E$-module to a coproduct $\bigvee_{n,\alpha\in B^n} \Sigma^n E$.*
2. *The function spectrum $\mathrm{Map}(X, E)$ is equivalent to a product $\prod_{n,\alpha\in B^n} \Sigma^{-n} E$.*
3. *We have isomorphisms $E_*(X) \simeq \pi_*(E) \otimes H_*(X)$ and $E^*(X) \simeq \mathrm{Hom}(H_*(X), \pi_*(E))$.*

*Proof.* It suffices to prove the first claim, since the second follows from the first from the following argument:

$$\begin{aligned}
&\mathrm{Map}(X, E)\\
\simeq &\mathrm{Map}_E(X \wedge E, E)\\
\simeq &\mathrm{Map}_E(\bigvee_{n,\alpha\in B_n} \Sigma^n E, E)\\
\simeq &\prod_{n,\alpha\in B_n} \mathrm{Map}(\Sigma^n E, E)\\
\simeq &\prod_{n,\alpha\in B_n} \Sigma^{-n} E
\end{aligned}$$

and the third is a direct consequence of the first two.
We identify $X$ and its suspension spectrum $\Sigma^\infty X_+$. Now, $X$ is connective and has free homology. We construct a sequence of spectra $X_0 \to X_1 \to \cdots$ having colimit $X$ with the following additional properties:

1. The map $X_n \to X$ induces an isomorphism in homology of degree $\le n$.
2. $X_n$ is built from finitely many spheres of dimension $\le n$. This implies that the cohomology $H^i(X_n; \mathbb{Z})$ vanish for $i \ge n$.

We inductively construct $X_n$. We let $X_{-1} = *$. Then suppose $X_{n-1}$ has been constructed. We denote $Y_n$ the cofiber of the map $X_{n-1} \to X$. Thus $Y_n$ is $(n-1)$-connected, and we have $H_n(X) \to H_n(Y_n)$ is an isomorphism. Thus by the Hurewicz theorem, the image of the homology classes $h_{n,\alpha}$ in $H_n(Y_n)$ are represented by maps $S^n \to Y_n$. We let $Z_n = \bigvee_{\alpha\in B_n} S^n$, and let $\varphi_n : Z_n \to Y_n$ be the induced map. Then we let the cofiber of $\varphi_n$ be $W_n$. Finally, we let $X_n$ be the fiber of the composition $X \to Y_n \to W_n$ (or equivalently, let $X_n = X \times_{Y_n} Z_n$). Now the above properties hold obviously ($X$ has homology in all dimensions, $Y_n$ has homology $\ge n$, $Z_n$ has homology concentrated at $n$, hence $W_n$ has homology $> n$).
Now we assume the first assertion holds. We prove the map $\theta : \bigvee_{n,\alpha\in B_n} \Sigma^n E \to E \wedge X$ is $k$-connected for all $k$. Notice that both spectra are connective, so we have $\theta$ 0-connected. Notice by construction that $Y_0 = X$. Therefore $\varphi_0$ induces an $E$-module map $\varphi_0 \wedge E : \bigvee_{\alpha\in B_0} E \simeq E \wedge Z_0 \to E \wedge X$. This map classifies homology classes in $E_0(X)$, say $b_{0,\alpha}$ by definition. Then, since $X$ is connective, we have $(\tau_{\ge 1}E)_0(X) = 0$, hence the map $E_0(X) \to (\tau_{\le 0}E)_0(X)$ is injective. Therefore $h''_{0,\alpha}$ is uniquely determined in $E_0(X)$. Notice then that $b_{0,\alpha}$ also restricts to $h'_{0,\alpha}$, hence $b_{0,\alpha} = h''_{0,\alpha}$. Thus the following diagram commutes, and the upper and lower sequences are fiber sequences.

$$\begin{tikzcd}
\bigvee_{\alpha\in B_0} E \arrow[r] \arrow[d] & \bigvee_{n,\alpha\in B_n} \Sigma^n E \arrow[r] \arrow[d] & \bigvee_{n\ge 0,\alpha\in B_n} \Sigma^n E \arrow[d] \\
E \wedge Z_0 \arrow[r] & E \wedge X \simeq E \wedge Y_0 \arrow[r] & E \wedge W_0
\end{tikzcd}$$

By definition, the left map is a homotopy equivalence, hence to prove the middle map $k$-connective, it suffices to prove the right map is $k$-connective. This is by the inductive hypothesis applied to the connective spectrum $\Sigma^{-1}W_0$.
Now we suppose the second condition holds. We prove the map $E \wedge X \to E \wedge Y_n$ admits a splitting $s_n : E \wedge Y_n \to E \wedge X$. Then since $E \wedge Y_n \to E \wedge X \to E \wedge Y_n$ is the identity, we have $E \wedge X \simeq (E \wedge Y_n) \vee (E \wedge X_{n-1})$. Also, consider the pullback $E \wedge X_n \simeq E \wedge X \times_{E\wedge Y_n} E \wedge Z_n$, which implies $E \wedge X_n \simeq (E \wedge Z_n) \vee (E \wedge X_{n-1})$. Then we have

$$E \wedge X \simeq \varinjlim(E \wedge X_n) \simeq \varinjlim \bigvee_{m\le n,\alpha\in B_m} \Sigma^m E \simeq \bigvee_{n,\alpha\in B_n} \Sigma^n E$$

We construct the splitting. Notice the cohomology classes $c''_{\alpha,n}$ induces a map

$$\psi_\alpha : Y_n \to E \wedge Y_n \to E \wedge X \to \Sigma^n E$$

Hence $\psi_\alpha \in E^n(Y_n)$. Notice that again by definition, we have the image of $\psi_\alpha$ in $(\tau_{\le 0}E)^n(Y_n)$ is $c'_{n,\alpha}$. Then $\psi_\alpha$ determines a map $\psi : Y_n \to \bigvee_\alpha \Sigma^n E \simeq E \wedge Z_n$. Moreover, the compatibility with $c'_{n,\alpha}$ shows that the composition $\psi \circ \varphi : E \wedge Z_n \to E \wedge Y_n \to E \wedge Z_n$ is the identity. Thus $Z_n$ splits off in $X$, hence, inductively, $Y_n$ splits off. □

**Corollary 9.2.3.** *For any complex oriented ring spectrum $E$, we have $E^*(\mathbb{CP}^\infty) = (\pi_*(E))[[t]]$, where $t$ has degree 2.*

*Proof.* Consider $E^*(\mathbb{CP}^n)$. Suppose $t \in E^*(\mathbb{CP}^n)$ is the restriction of the complex orientation of $E$ under the inclusion $\mathbb{CP}^n \hookrightarrow \mathbb{CP}^\infty$, then the cohomology class $\{1, t, \dots, t^n\}$ satisfies the second condition in theorem 9.2.2. Thus we have $\{1, t, \dots, t^n\}$ form a basis for $E^*(\mathbb{CP}^n)$. We then claim that $t^{n+1} = 0$, so that $E^*(\mathbb{CP}^n) \simeq (\pi_*(E))[t]/(t^{n+1})$. This is direct by taking the connective cover of $E$. Therefore by letting $\mathbb{CP}^\infty = \varinjlim \mathbb{CP}^n$, we obtain the result directly (notice the Mittag-Leffler condition holds here so that the $\varprojlim^1$ term vanishes). □

**Corollary 9.2.4.** *For a complex oriented ring spectrum $E$, we have $E^*(BU(n)) = (\pi_*(E))[[c_1, \dots, c_n]]$, where $c_i$ has degree $2i$.*

*Proof.* We first recall the analogous statement for ordinary cohomology in theorem 3.2.7. Then suppose $E_*(BU(1))$ has dual basis $\beta_i$ for $t^i$, and by theorem 9.2.2, we have $E_*(BU(n))$ freely generated by the classes $\{\beta_{i_1} \cdots \beta_{i_n}\}$, where these are the image of $\beta_{i_1} \times \cdots \times \beta_{i_n}$ under the map classifying the universal direct sum bundle $\theta : BU(1)^n \to BU(n)$. Then by the diagonal map $BU(n) \to BU(n)^2$, each $E_*(BU(n))$ is a coalgebra. For $n = 1$, this coalgebra structure is determined by the graded ring structure on $E^*(BU(1))$, namely, $\beta_n \mapsto \sum_{i+j=n} \beta_i \otimes \beta_j$. Then, $\theta$ determines the coalgebra structure on $E_*(BU(n))$ completely, and the coalgebra structure is given by the same formulas as in the case of integral homology. This implies that the multiplication in cohomology is defined as the same multiplication as in ordinary cohomology. This proves the claim. □

**Construction 9.2.5.** For a complex oriented ring spectrum $E$, and an $n$-dimensional complex vector bundle $\zeta$ over a space $X$, we define its $i$-th Chern class ($1 \le i \le n$) to be the pullback along the classifying map of $\zeta$ of $c_i \in E^{2i}(BU(n))$. The Chern classes satisfies the direct sum formula, as in the classical case. For bundles $\zeta$ and $\zeta'$ over $X$, we have

$$c_n(\zeta \oplus \zeta') = \sum_{i+j=n} c_i(\zeta)c_j(\zeta')$$

**Construction 9.2.6.** Let $E$ be a complex-oriented ring spectrum. Then notice by corollary 9.2.3, we have $E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \simeq (\pi_*(E))[[x, y]]$. Thus let $R = \bigoplus_n \pi_{2n}E$ be a graded ring, then let $\eta_1^1 \otimes \eta_2^1$ denote the tensor product of the pullback of the two universal line bundles.

This determines a map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$, which in turn induces a map $(\pi_*(E))[[t]] \to (\pi_*(E))[[x,y]]$, and the image of $t$ is, say $f$. Thus $f$ is a formal group law on $R$, compatible with the grading.
Equivalently, this formal group law characterizes the first Chern class of the tensor product of line bundles.

Interestingly, we have another characterization of the complex orientability of ring spectra.

**Definition 9.2.7.** Let $\zeta$ be a vector bundle of rank $n$, and $E$ a ring spectrum. Then we define an $E$-orientation of $\zeta$ is a cohomology class $u \in \tilde{E}^n(\mathrm{Th}(\zeta)) \simeq E^n(B(\zeta), S(\zeta))$ (see theorem 1.3.2) such that for each point $x \in X$, the restriction $x^*(u) \in E^n(B(\zeta|_x), S(\zeta|x)) \simeq \tilde{E}^n(\mathrm{Th}(\zeta|_x)) \simeq \tilde{E}^n(S^n) \simeq E^0(*)$ is a generator of $E^*(*) = \pi_*(E)$. Then we say $u$ is a Thom class or fundamental class for $\zeta$.

**Proposition 9.2.8.** *For a ring spectrum $E$ and an $E$-oriented vector bundle $\zeta$, we have $E^*(X) \simeq E^{*+n}(B(\zeta), S(\zeta))$, in which the map is given by multiplication by $u$. Similarly, we have $E_{*+n}(B(\zeta), S(\zeta)) \simeq E_*(X)$ given by the slant product with $u$.*

*Proof.* Directly follows from the Atiyah-Hirzebruch spectral sequence of the pair $E^*(B(\zeta), S(\zeta))$, in which the second page isomorphic to the second page of the spectral sequence of $E^*(X)$. □

**Theorem 9.2.9.** *A ring spectrum $E$ is complex orientable iff all complex vector bundles over any space is $E$-orientable (i.e. has a Thom class).*

*Proof.* It suffices to prove that $E$ is complex orientable iff all the universal bundles are $E$-orientable.
$\Longrightarrow$ : Recall from the proof of theorem 3.2.7 that $S(\eta^n) \simeq BU(n-1)$. Also recall that $E^*(BU(n)) \simeq (\pi_*(E))[[c_1, \ldots, c_n]]$. It follows that the map $\iota^* : E^*(BU(n)) \to E^*(BU(n-1))$ is surjective with kernel $(c_n)$. Thus $E^*(BU(n), BU(n-1))$ could be identified with the kernel. We claim that $c_n \in E^2n(BU(n), BU(n-1))$ is a Thom class. Since $BU(n)$ is connected, it suffices to check at any point the condition of definition 9.2.7. We consider the universal direct sum bundle on $BU(1)^n$. Under the classifying map $\theta$, $c_n \in E^2n(BU(n))$ is mapped to $t_1 \cdots t_n \in E^2n(BU(1)^n) \simeq (\pi_*(E))[[t_1, \ldots, t_n]]$. Therefore, $c_n \in E^{2n}(BU(n), BU(n-1))$ is mapped to

$$t_1 \cdots t_n \in E^2n(B((\eta^1)^n), S((\eta^1)^n)) = (t_1 \cdots t_n)(\pi_*E)[[t_1, \ldots, t_n]]$$

Now, if each $t_i$ restricts to a unit in $E^2(B(\eta^1|_x), S(\eta^1|_x))$, then clearly $t_1 \cdots t_n$ restricts to a unit at any point. Thus it suffices to prove for the claim for $n = 1$. But it follows by definition, since $c_1 \in \tilde{E}^2(\mathbb{CP}^\infty)$ restricts to $1 \in \tilde{E}^2(S^2) = \pi_0 E$ by definition.
$\Longleftarrow$ : Notice the Thom space of $\eta^1$ is homotopy equivalent to $\mathbb{CP}^\infty$. Therefore the Thom class of $\eta^1$ is the complex orientation. □

**Corollary 9.2.10.** *For a complex oriented ring spectrum $E$ and complex vector bundles $\zeta$ and $\zeta'$ of dimension $a$ and $b$ on $X$, suppose their Thom classes are given by $u_\zeta \in E^{2a}(B(\zeta), S(\zeta))$ and $u_{\zeta'} \in E^{2b}(B(\zeta'), S(\zeta'))$, we have the Thom class of $\zeta \oplus \zeta'$ given by*

$$(u_\zeta, u_{\zeta'}) \in \tilde{E}^{2a}(\mathrm{Th}(\zeta)) \oplus \tilde{E}^{2b}(\mathrm{Th}(\zeta')) \subset \tilde{E}^{2a+2b}(\mathrm{Th}(\zeta) \vee \mathrm{Th}(\zeta')) = \tilde{E}^{2a+2b}(\mathrm{Th}(\zeta \oplus \zeta'))$$

*Proof.* Since the Thom class is given by the top Chern class, this follows directly from the product formula. □

## 9.3 The complex coboridsm spectrum $MU$

**Definition 9.3.1.** We define $MU(n)$ to be the spectrum $\Sigma^{-2n}\operatorname{Th}(\eta^n) \simeq \Sigma^{-2n}(BU(n)/BU(n-1))$, where $\eta^n$ is the universal complex $n$-bundle on $BU(n)$, and we identify $\operatorname{Th}(\eta^n)$ with its suspension spectrum. Recall that we have for any real bundle $\zeta$ on $X$, we have $\Sigma\operatorname{Th}(\zeta) \simeq \operatorname{Th}(\zeta \oplus \varepsilon^1)$ where $\varepsilon^1$ is the trivial bundle of dimension 1. Thus we have a map $MU(n) \to MU(n+1)$ defined by

$$MU(n) \simeq \Sigma^{-2n}\operatorname{Th}(\eta^n) \simeq \Sigma^{-2n-2}\operatorname{Th}(\eta^n \oplus \varepsilon^1_{\mathbb{C}}) \to \Sigma^{-2n-2}\operatorname{Th}(\eta^{n+1}) = MU(n+1)$$

We define $MU$ to be the colimit of the sequence $MU(1) \to MU(2) \to \cdots$.

We have the following basic properties of $MU$.

**Lemma 9.3.2.** *$MU(0) \simeq S$, the sphere spectrum. $MU(1) \simeq \Sigma^{-2}\mathbb{CP}^\infty$. Moreover, the inclusion $MU(0) \to MU(1)$ is given by*

$$S = \Sigma^{-2}\Sigma^2 S \hookrightarrow \Sigma^{-2}\mathbb{CP}^\infty = MU(1)$$

*Proof.* $MU(0)$ is by definition the suspension spectrum of the Thom space of the trivial bundle on one point, hence is the suspension spectrum of $S^0$. It is easy to see that $\operatorname{Th}(\eta^1) \simeq \mathbb{CP}^\infty$, hence we have the second claim. The third claim is by the construction directly. □

**Lemma 9.3.3.** *$MU$ is a ring spectrum.*

*Proof.* Notice we have a map $m_{a,b} : BU(a) \times BU(b) \to BU(a+b)$ classifying the direct sum bundle. Passing to Thom spaces we have maps $MU(a)\wedge MU(b) \to MU(a+b)$ (notice the shifting cancels out directly). Then by passing to colimits, we obtain a map $MU \wedge MU \to MU$. □

**Lemma 9.3.4.** *The map $MU(n) \to MU(n+1)$ is given by the following composition.*

$$MU(n) \simeq MU(n) \wedge MU(0) \to MU(n) \wedge MU(1) \to MU(n+1)$$

*Where the last map is the map induced by $BU(n) \wedge BU(1) \to BU(n+1)$.*

*Proof.* Follows from the construction. □

**Lemma 9.3.5.** *$MU$ is complex oriented.*

*Proof.* Notice the map $\mathbb{CP}^\infty \simeq MU(1) \to MU$ represents a cohomology class $t \in \widetilde{MU}^2(\mathbb{CP}^\infty)$. This map restricts to a unit on $\widetilde{MU}^2(S^2)$ by the natural map $S^0 \simeq MU(0) \to MU(1)$, since $MU(0) \to MU$ is the unit of the ring spectrum. □

*Remark* 13. In fact, we may also define $MU$ using the following method. We let $MU(n)$ denote the Thom space of the universal bundle on $BU(n)$. Then we let $MU$ be the spectrum associated to the prespectrum $MU(0), MU(1), \ldots$.

Having the "$n$-th space" $MU(n) \simeq BU(n)/BU(n-1)$, we may see that $MU$ is closely related to orientations on complex vector bundles. In fact, we have the following theorem.

**Theorem 9.3.6.** *$MU$ is the universal complex oriented cohomology theory. That is, for every ring spectrum $E$ with a chosen orientation $t \in \tilde{E}^2(\mathbb{CP}^\infty)$, there exists a unique map $\varphi : MU \to E$, such that for a fixed element $t_{univ} \in \widetilde{MU}^2(\mathbb{CP}^\infty)$, $\varphi_*(t_{univ}) = t$.*

*Proof.* We let $E$ be a complex oriented cohomology theory with Thom classes $u_n \in \tilde{E}^{2n}(\operatorname{Th}(\eta^n)) \simeq \tilde{E}^{2n}(BU(n)/BU(n-1))$. Then we have maps $\varphi_n : MU(n) \to E$ classifying $u_n$. We claim the $\varphi_n$ satisfies the following properties.

1. The map $\varphi_1 : MU(1) \to E$ is given by the complex orientation of $E$.
2. The following diagram commutes up to homotopy.

$$\begin{tikzcd} MU(a) \wedge MU(b) \arrow[r] \arrow[d, "\varphi_a \wedge \varphi_b"'] & MU(a+b) \arrow[d, "\varphi_{a+b}"] \\ E \wedge E \arrow[r] & E \end{tikzcd}$$

3. The map $MU(n) \to MU(n+1) \to E$ coincides with $\varphi_n$.

The first claim is by definition. The second claim is by corollary 9.2.10. For the third claim, we consider lemma 9.3.4, and obtain that the composition $MU(n) \to MU(n+1) \to E$ is given by

$$MU(n) \simeq MU(n) \wedge MU(0) \to MU(n) \wedge MU(1) \to E$$

where the last map is given by the smash product of $\varphi_n$ and $\varphi_1$ followed by the product map of $E$. Thus this is equal to $\varphi_n$ by the second claim and that $\varphi_1$ restricts to $\varphi_0$.
By definition, $\mathrm{Map}(MU, E) \simeq \mathrm{Map}(\varinjlim MU(n), E) \simeq \varprojlim \mathrm{Map}(MU(n), E)$. Therefore we have a Milnor long exact sequence

$$\varprojlim{}^1 E^{-1}(MU(n)) \to E^0(MU) \to \varprojlim E^0(MU(n)) \to \varprojlim{}^1 E^0(MU(n))$$

The $\varprojlim^1$ terms vanish by the Mittag-Leffler condition (notice $E^*(MU(n+1)) \to E^*(MU(n))$ is surjective since it is the process of killing $c_{n+1}$ under a shift). Therefore $\varphi_n$ uniquely determine $\varphi : MU \to E$.
We then show that $\varphi$ is a map of ring spectra, that is, the following diagram commutes.

$$\begin{tikzcd} MU \wedge MU \arrow[r] \arrow[d] & MU \arrow[d] \\ E \wedge E \arrow[r] & E \end{tikzcd}$$

But since we have $E^0(MU) \simeq \varprojlim E^0(MU(n))$, this diagram commutes iff all diagrams in claim (2) commutes. Thus $\varphi$ is a map of ring spectra.
Finally we are left to show that $\varphi$ carries the complex orientation of $MU$ to the complex orientation of $E$. However this is by definition of $\varphi_1$. □

Now, as we have shown in construction 9.2.6, a complex orientation on $E$ induces a formal group law on $\bigoplus \pi_{2n}(E)$. As the following theorem presents, the universal complex oriented spectrum classifies the universal formal group law.

**Theorem 9.3.7.** *The map $\varphi : L \to \pi_*(MU)$ classifying is an isomorphism. Here $L$ is the Lazard ring.*

This will be the main topic of the two following sections. But now, we first calculate the homology of $MU$, which is much easier.

**Theorem 9.3.8.** *For any complex oriented cohomology theory $E$, we have $E_*(MU) \simeq (\pi_*(E))[b_1, b_2, \ldots]$, where $b_i$ has degree $2i$.*

*Proof.* Since $MU = \varinjlim MU(n)$, we have $E_*(MU) = \varinjlim E_*(MU(n))$. By proposition 9.2.8, we have $E_*(MU(n)) \simeq E_*(BU(n))$. Then recall $E_*(BU(n)) \simeq \mathrm{Sym}^n(\pi_*(E)\beta_0, \beta_1, \ldots)$, where $\beta_i$ is the dual basis of $t^i$ for $E^*(BU(1))$. Then we have $E_*(MU(n)) \simeq \mathrm{Sym}^n(\pi_*(E))b_0, b_1, \ldots$, where $b_i$ is the dual basis of $t^{i+1}$. Notice that the map $E_*(MU(n)) \to E_*(MU(n+1))$ is given by multiplying with $b_0$. Then finally notice that we may identify $\mathrm{Sym}^n(\pi_*(E))b_0, b_1, \ldots$ with $\bigoplus i \leq n\mathrm{Sym}^i(\pi_*(E))b_1, \ldots$ by mapping $b_0$ to 1. Then by passing to colimits we obtain $E_*(MU) \simeq (\pi_*(E))[b_1, b_2, \ldots]$. □

*Remark* 14. This is analogous to the calculation of $H_*(MO)$.

In particular, we have $H_*(MO) \simeq L$, the Lazard ring.

**Construction 9.3.9.** Notice that for any complex oriented cohomology theory $E$, the smash product $MU \wedge E$ has two complex orientations, associated to the pushforward of the orientation of $MU$ and $E$ respectively. Notice $\pi_*(MU \wedge E) \simeq E_*(MU) \simeq (\pi_*(E))[b_1, b_2, \dots]$. Explicitly, we have two classes $t_{MU}, t_E \in \widetilde{MU \wedge E}^2(\mathbb{CP}^\infty)$ such that

$$(\pi_*(E))[b_1, b_2, \dots][[t_E]] \simeq (MU \wedge E)^*(\mathbb{CP}^\infty) \simeq (\pi_*(E))[b_1, b_2, \dots][[t_{MU}]]$$

Therefore we may express $t_{MU}$ as a power series $t_{MU} = \sum_{i \geq 1} a_i t_E^i$.

**Proposition 9.3.10.** *We have* $t_{MU} = t_E + b_1 t_E^2 + \cdots$.

*Proof.* We first recall that a class in $\widetilde{MU \wedge E}^2(\mathbb{CP}^\infty)$ can be represented by a map $MU(1) \to MU \wedge E$. Equivalently, we may consider a $E$-module map $MU(1) \wedge E \to MU \wedge E$. We suppose $t_M U$ and $t_E$ are represented by maps $\varphi_{MU}$ and $\varphi_E$. Then, notice $b_i \in E_{2i}(MU(1))$ determines a map $\Sigma^{2i} E \to MU(1) \wedge E$. By theorem 9.2.2, we have an equivalence of $E$-module spectra $\bigvee_{i \geq 0} \Sigma^{2i} E \simeq MU(1) \wedge E$. Thus, to specify a map from $MU(1) \wedge E$ to $MU \wedge E$, it suffices to specify its restriction to $\Sigma^{2i} E$ for all $i$.
Let the map $\lambda : E \wedge MU(1) \to E$ be the complex orientation on $E$, then we have $\varphi_E$ the composition $E \wedge MU(1) \to E \to E \wedge MU$. By construction, $\lambda$ vanishes on $\Sigma^{2i} E$ for $i > 0$ and restricts to the identity when $i = 0$. Then, the map $\varphi_{MU}$ is given by smashing with $E$ the map $MU(1) \to MU$. Thus $\varphi_{MU}$ is the coproduct of the maps $\varphi^i_{MU} : \Sigma^{2i} E \to MU \wedge E$ classified by $b_i$.
Finally, notice that the complex orientation $t_E$ induces a map $\Sigma^{-2}(MU(1) \wedge E) \to MU(1) \wedge E$ by the slant product. This action translates $MU(1) \otimes E$ by one degree downward. Therefore $\varphi^i_{MU}$ can be identified with the map translated downward $i$ times composed with $b_i \varphi_E$. Finally, we return to $t_{MU}$ and $t_E$ and obtain $t_{MU} = \sum b_i t_E^{i+1}$. ☐

Then, recall from construction 9.2.6 that every complex orientation induces a formal group law on $R = \pi_*(MU \wedge E) = (\pi_*(E))[b_1, b_2, \dots]$. By the above proposition we clearly have the following.

**Corollary 9.3.11.** *Let* $R = (\pi_*(E))[b_1, b_2, \dots]$*, and* $f_E$*,* $f_{MU}$ *be the formal group laws associated to* $t_E$ *and* $t_{MU}$*. Then let* $g(x) \in R[[x]]$ *be* $x + b_1 x^2 + \cdots$*, we have* $t_{MU} = t_E$*. Hence we have*

$$f_{MU}(x, y) = g \circ f_E(g^{-1}(x), g^{-1}(y))$$

Then, by restricting to $H\mathbb{Z}$, we get the following.

**Corollary 9.3.12.** *Let* $E = MU \wedge H\mathbb{Z}$*, equipped with the complex orientation coming from* $MU$*. Then* $\pi_*(E) = \mathbb{Z}[b_1, \dots]$*, and the formal group law is given by* $f(x, y) = g(g^{-1}(x) + g^{-1}(y))$*, where* $g = x + b_1 x^2 + \cdots$.

*Proof.* Recall the formal group law on $H\mathbb{Z}$ is given by addition. ☐

Notice the map $L \to H_*(MU; \mathbb{Z})$ is exactly the same as in theorem 9.1.4, thus by theorem 9.1.4, we have the composition $L \to \pi_*(MU) \to H_*(MU; \mathbb{Z})$ is a rational isomorphism, where the second map is the Hurewicz map. Recall that the Hurewicz map is always an rational isomorphism, hence we have the following.

**Corollary 9.3.13.** *The map* $L \otimes \mathbb{Q} \to \pi_*(MU) \otimes \mathbb{Q}$ *is an isomorphism.*

Thus to prove the theorem $L \simeq \pi_*(MU)$, it suffices to prove that they are isomorphic after $p$-adic completion for all $p$ (this is an purely algebraic result). We will use the Adams spectral sequence for $MU$ to prove this result.

## 9.4 The Adams spectral sequence for $MU$

We first recall some basic facts about the Adams spectral sequence.

**Construction 9.4.1.** Let $R = H\mathbb{F}_p$ for some prime $p$. Then we have a cosimplicial object $R^\bullet$ in **Sp**, given by

$$[n] \mapsto R \wedge \cdots \wedge R = R^{\wedge n+1}$$

and the boundary and degeneracy maps the obvious ones. Moreover, we may extend this cosimplicial object to a augmented simplicial object $\overline{R}^\bullet$ by letting $\overline{R}^{-1} = S$, the sphere spectrum. For any other spectrum $X$, we let $\overline{X}^\bullet = X \wedge \overline{R}^\bullet$, and $X^\bullet = X \wedge R^\bullet$. Then, we have a map $X = \overline{X}^{-1} \to \varprojlim X^\bullet$. Finally, we let $E_1^{a,b} = \pi_a(X^b) = \pi_a(X \wedge R^{\wedge b+1})$, with the differential given on the first page to be

$$\pi_a(X^0) \xrightarrow{d^0 - d^1} \pi_a(X^1) \xrightarrow{d^0 - d^1 + d^2} \cdots$$

For simplicity, we denote $\varprojlim X^\bullet$ by $\mathrm{Tot}(X^\bullet)$ and the partial limit $\varprojlim_{1 \le i \le n} X^\bullet$ by $\mathrm{Tot}_n(X^\bullet)$. Notice that in general context, the totalization of a simplicial object is defined via ends, and the author is not clear of the difference of the two notions.
We have the following theorem.

**Theorem 9.4.2.** *Suppose $X$ is a connective spectrum with $\pi_n(X)$ finitely generated for all $n$. Then the map $X \to \mathrm{Tot}\, X^\bullet$ has the following properties.*

1. *For all $n$, we have the following filtration*

$$\cdots \subseteq F^2\pi_n(X) \subseteq F^1\pi_n(X) \subseteq F^0\pi_n(X) \simeq \pi_n(X)$$

   *Where $F^i\pi_n(X)$ is the kernel of $\pi_n(X) \to \pi_n(\mathrm{Tot}_i\, X^\bullet)$.*

2. *For each $i \ge 0$, there is some $j > i$ such that $(p^i)\pi_n(X) \subseteq F^j\pi_n(X) \subset (p^j)\pi_n(X)$. In particular, we have $\varprojlim(\pi_n(X)/F^j\pi_n(X)) \simeq (\pi_n(X))_p$, the $p$-adic completion.*

3. *As in the case for general spectral sequences, for fixed $a$ and $b$, we have $E_r^{a,b}$ stablize to $E_\infty^{a,b}$ for $r \gg 0$. Moreover, we have $E_\infty^{a,b} \simeq F^b\pi_{a+b}(X)/F^{b+1}\pi_{a+b}(X)$. If $X$ is a ring spectrum, we have the following additional properties.*

4. *The filtration agrees with the multiplicative structure on $\pi_*(X)$. Explicitly, the multiplication map $\pi_m(X) \times \pi_n(X) \to \pi_{m+n}(X)$ restricts to a map $F^i\pi_m(X) \times F^j\pi_n(X) \to F^{i+j}\pi_{m+n}(X)$.*

5. *The spectral sequence is multiplicative. That is, for each $r$ we have maps $E_r^{a,b} \times E_r^{a',b'} \to E_r^{a+a',b+b'}$. These maps are associative and satisfies the Leibniz rule when applied the differential. Finally, the map $E_\infty^{a,b} \times E_\infty^{a',b'} \to E_\infty^{a+a',b+b'}$ agrees with the quotient map of $F^b\pi_{a+b}(X) \times F^{b'}\pi_{a'+b'}(X) \to F^{b+b'}\pi_{a+b+a'+b'}(X)$.*

**Construction 9.4.3.** Again let $R = H\mathbb{F}_p$. We recall that the Steenrod algebra is the Hopf algbera $\mathcal{A} \simeq H^*(R; \mathbb{F}_p)$. Then we have the dual Steenrod algebra $\mathcal{A}_* \simeq \pi_*(R \wedge R)$.
Also recall that we define for a ring spectrum $E$ and module spectrums $X$ and $Y$, we have the definition

$$X \wedge_E Y = \varinjlim \left( X \wedge Y \mathrel{\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}} X \wedge E \wedge Y \mathrel{\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}} \cdots \right)$$

Then, notice that we have

$$X^b = X \wedge R^{\wedge b+1} = (X \wedge R) \wedge_R (R \wedge R) \wedge_R \cdots$$

Therefore, we have $E_1^{a,b} \simeq H_a(X; \mathbb{F}_p) \otimes_{\mathbb{F}_p} (\mathcal{A}_*)^{\otimes b}$, and the differentials is given by the coalgebra structure of $\mathcal{A}_*$, and the comodule structure of $H_*(X; \mathbb{F}_p)$.

**Construction 9.4.4.** From the above construction, we fix $p = 2$ for convenience in applying algebro-geometric languages. let $\mathbb{G}$ denote the spectrum of the commutative ring $\mathcal{A}_*$, and the Hopf algebra structure on $\mathcal{A}_*$ induces a group scheme structure on $\mathbb{G}$. Therefore, we have $\mathbb{G}$ a group scheme over $\operatorname{Spec}\mathbb{F}_2$. Then let $V$ denote the $\mathbb{F}_2$-vector space $H_*(X; \mathbb{F}_2)$. The coaction $V \to V \otimes \mathcal{A}_*$ induces a structure of a representation of the algebraic group $\mathbb{G}$ on $V$. In this term, the $E_1^{*,*}$ term is simply given by the cohomology of the complex (with each term graded)

$$V \to \Gamma(\mathbb{G}; V \otimes \mathcal{O}_{\mathbb{G}}) \to \Gamma(\mathbb{G}^2; V \otimes \mathcal{O}_{\mathbb{G}^2}) \to \cdots$$

And therefore the $E_2^{*,b}$ term can be identified with $H^b(\mathbb{G}; H_*(X; \mathbb{F}_2))$.
If $X$ is a ring spectrum, then $H_*(X; \mathbb{F}_2)$ is itself a commutative $\mathbb{F}_2$-algebra, hence we may form the spectrum $Y = \operatorname{Spec}(H_*(X; \mathbb{F}_2))$. Therefore the action of $\mathbb{G}$ on $Y$ gives a quotient stack $Y/\mathbb{G}$, whose cohomology, by definition, can be identified with the $E_2$ page. Therefore we simply have $E_2^{*,*} = H^*(Y/\mathbb{G}; \mathcal{O}_{Y/\mathbb{G}})$ (with natural grading on $Y$).

**Construction 9.4.5.** We let $X(n) = \operatorname{Map}(\mathbb{RP}^n, S)$. Then $X(n)$ is an $\mathbb{E}_\infty$ ring spectrum, and we have $H_*(X(n); \mathbb{F}_2) \simeq H^*(\mathbb{RP}^n; \mathbb{F}_2) \simeq \mathbb{F}_2[x]/(x^{n+1})$. Then we pass through the limit and get $\varinjlim \operatorname{Spec} H_*(X(n); \mathbb{F}_2) \simeq \operatorname{Spf} H^*(\mathbb{RP}^\infty; \mathbb{F}_2) = \mathbb{F}_2[[x]]$. Then we naturally have an action of $\mathbb{G}$ on $\operatorname{Spf} H^*(\mathbb{RP}^\infty; \mathbb{F}_2)$. Notice that the map $\mathbb{RP}^\infty \times \mathbb{RP}^\infty \to \mathbb{RP}^\infty$ is the colimit of the maps $\mathbb{RP}^m \times \mathbb{RP}^n \to \mathbb{RP}^{m+n}$, and hence this induces a map

$$\operatorname{Spf}\mathbb{F}_2[[x]] \times_{\operatorname{Spec}\mathbb{F}_2} \operatorname{Spf}\mathbb{F}_2[[x]] \to \operatorname{Spf}\mathbb{F}_2[[x]]$$

given by a power series $f(x, y) = x + y$. By definition, $f$ is a formal group law over $\mathbb{F}_2$, and $\mathbb{G}$ respects the group structure given by $f$. Therefore the $\mathbb{G}$ action on $\operatorname{Spf}\mathbb{F}_2 \times_{\operatorname{Spec}\mathbb{F}_2} \operatorname{Spec} A = \operatorname{Spf} A[[x]]$ can be viewed as an automorphism of the formal group $x + y$. Therefore, the automorphism must be of the form $x \mapsto x + a_1x^2 + a_2x^4 + a_3x^8 + \cdots$ since they are the only series preserving addition in characteristic 2. Recall also the computation of the dual Steenrod algebra theorem 4.1.11. This implies that $\mathbb{G}$, viewed as a functor **Ring** $\to$ **Set**, is given by $\operatorname{Hom}(\operatorname{Spec} A, \mathbb{G})$ to be all power series

$$x \mapsto x + a_1x^2 + a_2x^4 + a_3x^8 + \cdots$$

Now, we apply the above constructions to the case where $X = MU$. Recall that we have $H_*(MU; \mathbb{F}_2) = \mathbb{F}_2[b_i]$ for $b_i$ of degree $2i$. We let $Z$ denote the spectrum $\operatorname{Spec} H_*(MU; \mathbb{F}_2)$, then for any $\mathbb{F}_2$ algebra $R$, the map $\operatorname{Hom}(\operatorname{Spec} R, Z)$ can be identified with the set consisting of all power series $y + b_1y^2 + b_2y^3 + \cdots$. Therefore we see that $Z$ classifies the coordinates of the additive formal group on $R$ preserving the first order terms. By the description of $\mathbb{G}$ as the group of automorphism of the additive formal group, we have a natural action of $\mathbb{G}$ on $Z$.

However, we notice that this action is different to the desired action in the Adams spectral sequence of $MU$, due to the problems of cohomological gradings. We now construct the correct action.

**Construction 9.4.6.** For a complex oriented ring spectrum $E$ where $\pi_*(E)$ is a $\mathbb{F}_2$ algebra, notice that $\operatorname{Hom}(\mathcal{A}_*, \pi_*(E))$ (respect gradings) is the group of automorphisms on the additive formal group on $\pi_*(E)$. But the ring $\operatorname{Hom}(H_*(MU; \mathbb{F}_2), \pi_*(E))$ is the coordinates of the additive formal group on $\bigoplus_i \pi_{2i}(E)$. This implies that the correct action of $\mathbb{G}$ on $Z$ should differ by our construction with a Frobenius map $F : \mathbb{G} \to \mathbb{G}$. Explicitly, consider $\mathbb{G}' \subseteq \mathbb{G}$ be the image of the Frobenius map. Then for any ring $R$, we define the $R$-point $f = x + a_1x^2 + \cdots$ of $\mathbb{G}'$ acts on the $R$-points of $Z$ by $x \mapsto x + a_1^2x^2 + \cdots$. Finally we define the action of $\mathbb{G}$ on $Z$ to be the action factoring through the Frobenius map.

**Lemma 9.4.7.** *The action described above is indeed the action induced by the $\mathcal{A}_*$ coalgebra structure of $H_*(MU; \mathbb{F}_2)$.*

*Proof.* This can be done by direct computation of the coalgebra structure of $H_*(\mathbb{CP}^\infty; \mathbb{F}_2)$ (if someone sees this, remind me to finish the computation). □

We first calculate the cohomology of $Z/\mathbb{G}'$.

**Lemma 9.4.8.** *Let $Z_0 = \operatorname{Spec} \mathbb{F}_2[b_2, b_4, \dots]$ (with generators of degree not of the form $2^r - 1$), then the action of $\mathbb{G}'$ on $Z$ determines an isomorphism $a : \mathbb{G}' \times Z_0 = Z$, that is, we have an isomorphism of schemes $Z/\mathbb{G}' = Z_0$.*

*Proof.* It suffices to prove that for all $R$, we have the following: for all $R$-points of $Z$, say $h(x) = x + c_1x^2 + \cdots$, $h$ can be uniquely factored into $f \circ g$, where $f$ is of the form $f(x) = x + a_1x^2 + a_2x^4 + \cdots$ and $g(x) = x + b_2x^3 + b_4x^5 + \cdots$. This is obvious. □

Correct Lurie!!!

Finally we see the following results.

**Theorem 9.4.9.** *We have the second page of the mod-2 Adams spectral sequence of $MU$ given by*

$$E_2^{*,*} \simeq \mathbb{F}_2[b_2, b_4, \dots, \varepsilon_1, \varepsilon_2, \dots]$$

*where $b_i$ $(i \neq 2^r - 1)$ has bidegree $(2i, 0)$, and $\varepsilon_j$ has bidegree $(2^j - 1, 1)$.*

Therefore, the Adams spectral sequence degenerates since there is no room for non-trivial maps. The mod-$p$ Adams spectral sequence behaves exactly the same as the mod-2 case, namely, we have

**Theorem 9.4.10.** *We have the second page of the mod-$p$ (for all prime $p$) Adams spectral sequence of $MU$ given by*

$$E_2^{*,*} \simeq \mathbb{F}_p[c_1, c_2, \dots]$$

*where $c_i = \varepsilon_j$ if $i + 1 = p^j$, and $c_i = b_i$ otherwise (again $b_i$ has bidegree $(2i, 0)$ and $\varepsilon_j$ has bidegree $(p^j - 1, 1)$).*

*Proof.* By essentially the same proof, but with super algebraic geometry instead since for odd $p$, $\mathcal{A}_*$ is supercommutative (maybe I will add an appendix). □

Therefore, by theorem 9.4.2, we deduce the following result.

**Corollary 9.4.11.** *The associated graded ring $\operatorname{gr}(\pi_*(MU))$ of the mod-$p$ Adams filtration on $\pi_*(MU)$ is isomorphic to a polynomial algebra $\mathbb{F}_p[c_0, \dots]$, where $c_i \in \operatorname{gr}_0(\pi_{2i}(MU))$ for $i+1 \neq p^j$ and $c_i \in \operatorname{gr}_1(\pi_{2i}(MU))$ for $i + 1 = p^j$.*

**Corollary 9.4.12.** $(\pi_*MU)^\vee = \mathbb{Z}_p[u_1, u_2, \dots]$. *Here $\mathbb{Z}_p$ denotes the $p$-adic completion of $\mathbb{Z}$.*

We are finally ready to prove theorem 9.3.7.

*Proof.* Recall that we have by lemma 9.1.6, the map $L \to \pi_*(MU) \to H_*(MU; \mathbb{Z})$ is injective. Thus it suffices to show that the map $L \to \pi_*(MU)$ is surjective. In fact, we show that it is surjective after $p$-adic completion for all $p$. By theorem 9.1.4, we have $L^\vee \simeq \mathbb{Z}_p[t_1, t_2, \dots]$, and we have by corollary 9.4.12 that $(\pi_*(MU))^\vee \simeq \mathbb{Z}_p[u_1, u_2, \dots]$. Let $I$, $J$ be defined as in lemma 9.1.6, and $K$ the likewise ideal for $(\pi_*(MU))^\vee$. As in the proof of theorem 9.1.4, it suffices to prove the map $I/I^2 \to K/K^2$ is surjective in each degree.
Recall that $(I/I^2)_{2n} = \mathbb{Z}_p t_n$ and $(K/K^2)_{2n} = \mathbb{Z}u_n$, hence $\theta(t_n) = \lambda u_n + c$, where $c$ is decomposable in $(\pi_*(MU))^\vee$. It suffices to prove that $\lambda$ is a $p$-adic unit. But then, the Hurewicz map maps $u_n$ to $\lambda' b_n + d$, where $d$ is again decomposable. Thus by theorem 9.1.4, $\lambda\lambda'$ is $p$ if $n+1 = p^r$ and 1 otherwise. If $n+1$ is not a power of $p$, then $\lambda$ is obviously a unit. If $n+1 = p^r$, it suffices to prove $\lambda'$ is divisable by $p$, which is equivalent to the image of $u_n$ vanishes in the ring $H_*(MU; \mathbb{F}_p)$. This is equivalent to the requirement that $u_n$ has Adams filtration degree $\geq 1$, which is true. □

Finally, we return to the Adams spectral sequence.

**Construction 9.4.13.** Recall in theorem 9.4.2, $R = H\mathbb{F}_p$ for some $p$. We now let $R = MU$. By the same construction we obtain a cosimplicial spectrum $X^\bullet$. Now, for connective $X$, the map $X \to \varprojlim X^\bullet \simeq \varprojlim_n (\varprojlim_{i \leq n} X^i)$ is an equivalence. We call the spectral sequence associated to this cosimplicial spectrum the Adams-Novikov spectral sequence.
Since $\pi_0(MU) = \pi_0(S) = Z$, the convergence is faster than the mod-$p$ Adams spectral sequence. In fact, if we denote $F^n\pi_* X$ to be the kernel of $\pi_* X \to \pi_* \varprojlim_{i \leq n+1} X^\bullet$, for each $n$ we have a finite filtration

$$0 = F^{n+1}\pi_n(X) \subseteq F^n\pi_n(X) \subseteq \cdots \subseteq F^0\pi_n(X) = \pi_n(F)$$

**Construction 9.4.14.** Again, like the mod-$p$ Adams spectral sequence, the chain complex

$$MU_*(X) \to (MU \wedge MU)_*(X) \to (MU \wedge MU \wedge MU)_* X \to \cdots$$

has an algebro-geometric interpretation. Recall $\pi_*(E \wedge MU) = E_*(MU) = \pi_*(E)[b_1, b_2, \ldots]$, for all complex oriented cohomology theory $E$. Then, letting $\pi_*(E) = R$, $\operatorname{Spec} \pi_*(E \wedge MU)$ is an infinite dimensional affine space over $\operatorname{Spec} R$. Explicitly, it is the affine space parametrizing all coordinates $g(t) = t + b_1 t^2 + \cdots$ on $R[[t]]$ agreeing with the standard coordinate to the first order.
Then, we let $G = \operatorname{Spec} \mathbb{Z}[b_1, b_2, \ldots]$. This is a group scheme, whose group structure on $R$-points (namely, power series of the form $g(t) = t + b_1 t^2 + \cdots$) is given by composition of power series. Therefore, we have $\operatorname{Spec} \pi_*(E \wedge MU) = G \times \operatorname{Spec} \pi_*(E)$. In particular, we have $\operatorname{Spec} \pi_*(MU \wedge MU) = G \times \operatorname{Spec} L$. The ring structure of $MU$ encodes a $G$ action on $\operatorname{Spec} L$, namely, a power series $g$ acts on $\operatorname{Spec} L$ by sending the formal group law $f(x, y)$ to $g(f(g^{-1}(x), g^{-1}(y)))$.

**Definition 9.4.15.** We define $\mathcal{M}^s_{FG}$ to be the quotient stack $\operatorname{Spec} L/G$. Equivalently, this is the moduli stack sending a ring $R$ to the groupoid of all formal group laws with strict isomorphisms. Note that this is indeed a stack w.r.t the flat topology.

**Construction 9.4.16.** Let $X$ be any spectrum. Then, $\pi_*(X^n) = \pi_*(X \wedge MU^{\wedge n+1})$ is a module over the commutative ring $\pi_*(MU^{\wedge n+1})$, hence can be identified with a quasi-coherent sheaf on $\operatorname{Spec} L \times G^n$. Notice that the quasi-coherent sheaves $\pi_*(X^n)$ are compatible with one another, hence they form a quasi-coherent sheaf over $\mathcal{M}^s_{FG}$. We call this quasi-coherent sheaf $\mathcal{F}_X$. Moreover, the cochain complex

$$MU_*(X) \to (MU \wedge MU)_*(X) \to \cdots$$

is the standard complex for computing the cohomology of $\mathcal{M}^s_{FG}$ with $\mathcal{F}_X$ coefficients. Thus we have the following.

**Theorem 9.4.17.** *For any spectrum $X$, the second page of the Adams-Novikov spectral sequence of $X$ is given by*

$$E_2^{*,b} \simeq H^b(\mathcal{M}^s_{FG}; \mathcal{F}_X) = H^b(G; MU_*(X))$$

We will focus on this point of view in the next chapter.

# Chapter 10

# The moduli stack of formal groups

## 10.1 The moduli stack $\mathcal{M}_{FG}$

**Definition 10.1.1.** Besides the $G$ action on $\operatorname{Spec} L$, we have an action of the multiplicative group scheme $\mathbb{G}_m$ on $\operatorname{Spec} L$, given by $f(x, y) \mapsto \lambda f(\lambda^{-1}x, \lambda^{-1}y)$. We define the group scheme $G^+$ to be the semidirect product of $G$ and $\mathbb{G}_m$, whose $R$-points are given by series $g(x) = b_0x + b_1x^2 + \cdots$, where $\lambda$ is invertible. $g$ has an action on the formal group laws, given by $g(f(g^{-1}(x), g^{-1}(y)))$. These kinds of isomorphisms between formal group laws are called isomorphisms (in contrast to strict isomorphisms), and we define $\mathcal{M}_{FG} = \operatorname{Spec} L/G^+$ to be the moduli stack of formal group laws with isomorphisms.

Recall from the appendix that a formal group is simply a group object in the category of formal schemes. Therefore, we have the following definition.

**Definition 10.1.2.** For a formal group law $f \in R[[x, y]]$, we define $\mathcal{G}_f$ to be the formal group over $\operatorname{Spec} R$ whose $A$-points (here $A$ is an $R$-algebra) is given by

$$\mathcal{G}_f(A) = \{a \in A | \exists n, a^n = 0\}$$

and the group structure is given by $a \cdot b = f(a, b)$. We call $\mathcal{G}_f$ the formal group associated to $f$. We define a formal group over $R$ to be coordinatizable, iff it is of the form $\mathcal{G}_f$ for some $f$.

**Proposition 10.1.3.** *Isomorphisms between coordinatizable formal groups are given by isomorphisms between the corresponding formal group laws.*

*Proof.* Suppose we have $\alpha : \mathcal{G}_f \simeq \mathcal{G}_{f'}$ an isomorphism. Then, consider $\alpha_n = \alpha(R[t]/(t^{n+1}))$. Then $\alpha_n$ induces a map from nilpotent elements in $R[t]/(t^{n+1})$ to itself. We consider the element $\alpha_n(t) = b_0t + b_1t^2 + \cdots + b_{n-1}t^n$. Then by functorality of $\alpha$ and $\mathcal{G}_f$, we have $\alpha_{n+1}(t) = b_0t + \cdots + b_nt^{n+1}$. These elements put together to a formal power series $g(t) = b_0t + \cdots$. This is the isomorphism between $f$ and $f'$. $\square$

It is now important to make precise what the moduli stacks $\mathcal{M}^s_{FG}$ and $\mathcal{M}_{FG}$ represents. The main obstruction is that the functor assigning a ring $R$ to its coordinatizable formal group laws with isomorphisms does not satisfy descent. Therefore we need the following definition.

**Definition 10.1.4.** For a ring $R$, we define a formal group over $R$ to be a functor $\mathcal{G} : \mathrm{Alg}_R \to \mathbf{Ab}$ satisfying the two following properties.

1. $\mathcal{G}$ is a Zariski sheaf.

2. $\mathcal{G}$ is Zariski locally isomorphic to coordinatizable formal groups.

**Theorem 10.1.5.** *$\mathcal{M}_{FG}$ sends each ring $R$ to the groupoid of formal groups in the sense of definition 10.1.4 with isomorphisms as morphisms.*

*Proof.* This can be viewed as a stackification. Make this clear! ☐

**Definition 10.1.6.** For a ring $R$ and a formal group $\mathcal{G}$ over $R$, we define its Lie algebra $\mathfrak{g}$ to be the abelian group $\ker(\mathcal{G}(R[t]/(t^2)) \to \mathcal{G}(R))$. In fact, it is an $R$-module, with $\lambda \in R$ acting on $\mathfrak{g}$ by the endomorphism on $R[t]/(t^2)$ given by $t \mapsto \lambda t$.

**Lemma 10.1.7.** *For a coordinatizable formal group $\mathcal{G}_f$, we have its Lie algebra $\mathfrak{g} \simeq R$. Moreover, for $g(x) = b_0x + b_1x^2 + \cdots$, If $g$ is an endomorphism of $f$, then $g$ induces a map on the Lie algebra given by multiplication with $b_0$.*

*Proof.* Direct. ☐

**Theorem 10.1.8.** $\mathcal{M}^s_{FG}$ *represents the functor sending $R$ to all coordinatizable formal groups, with morphisms isomorphisms between the coordinatizable formal groups preserving the trivialization of the Lie algebra.*

*Proof.* By definition. ☐

**Proposition 10.1.9.** *A formal group $\mathcal{G}$ is coordinatizable iff $\mathfrak{g} \simeq R$.*

*Proof.* $\Longrightarrow$ : Obvious.
$\Longleftarrow$ : We only present an idea here. We fix an isomorphism $\mathfrak{g} \simeq R$. Since $\mathcal{G}$ is a formal group, we localize it by the second property, and we have $\mathcal{G}$ is coordinatizable on each open set $D(t_i)$, say isomorphic to $\mathcal{G}_{f_i}$. Notice that this identification makes $\mathfrak{g}$ a locally free module of rank 1 on $R$. Then, we attempt to glue all $\mathcal{G}_{f_i}$ together. Suppose that the transition between the restriction of $\mathcal{G}_{f_i}$ and $\mathcal{G}_{f_j}$ on $D(t_it_j)$ is given by $g_{i,j}(t) = b_0t + b_1t^2 + \cdots$. Then the transition between the restriction of $\mathfrak{g}_i$ and $\mathfrak{g}_j$ on $D(t_it_j)$ is exactly given by $b_0$, by lemma 10.1.7. Then, $\mathfrak{g}_i$ glueing to $\mathfrak{g}$ is isomorphic to $R$ iff all $b_0$ term agrees, which is also equivalent to the condition that all $\mathcal{G}_{f_i}$ glues together to some $\mathcal{G}_f$. Make this precise! ☐

Now we have seen that ordinary formal group laws equipped with strict isomorphisms corresponds to coordinatizable formal groups, hence to $\mathcal{M}^s_{FG}$. We now present a concept so as to make non-coordinatizable formal groups more concrete.

**Definition 10.1.10.** Let $R$ be a ring and $\mathcal{L}$ be an invertible $R$-module. An $\mathcal{L}$-twisted formal group law is a formal series $f(x, y) = \sum a_{i,j}x^iy^j$, where $a_{i,j} \in \mathcal{L}^{\otimes i+j-1}$, and satisfies the usual identities of formal group laws.

*Remark* 15. Here the degree $a_{i,j} \in \mathcal{L}^{\otimes i+j-1}$ is since this term behaves like an $(i + j - 1)$-fold tensor. Consider the isomorphism $g(t) = \lambda t$, then $a_{i,j}$ is multiplied by $\lambda^{i+j-1}$ times.

**Theorem 10.1.11.** *$\mathcal{L}$-twisted formal group laws corresponds to formal groups with their Lie algebra equal to $\mathcal{L}^{-1}$.*

*Proof.* Similar. ☐

*Remark* 16. Notice the notion of $\mathcal{L}$-twisted formal group laws is merely the formal group laws respecting gradings on the ring $\bigoplus_{n\in\mathbb{Z}} \mathcal{L}^{\otimes n}$, where $\mathcal{L}^{\otimes n}$ has grade $2n$. Therefore $\bigoplus \mathcal{L}^{\otimes n}$ is equipped with a natural action by the lazard ring $L$ given a $\mathcal{L}$-twisted formal group law $f$.

**Construction 10.1.12.** We may consider the assignment $(R, \mathcal{G}) \mapsto \mathfrak{g}^{-1}$ as a line bundle $\omega$ on the moduli stack $\mathcal{M}_{FG}$. With the above proposition 10.1.9, we see that $\mathcal{M}^s_{FG}$ is equivalently, the total space of $\omega$ with the zero section removed (in other words, the space of sections/trivializations of $\omega$). This gives a more precise version of theorem 9.4.17.

**Theorem 10.1.13.** *Let $X$ be a spectrum. Then notice $MU_{2*}(X)$ and $MU_{2*+1}(X)$ are both modules over $L = \pi_*(MU)$ carrying compatible actions of $G^+$, and therefore determine sheaves $\mathcal{F}_X^{\text{even}}$ and $\mathcal{F}_X^{\text{odd}}$. Then the second page of the Adams-Novikov spectral sequence is given by*

$$E_2^{2a,b} = H^b(\mathcal{M}_{FG}; \mathcal{F}_X^{\text{even}} \otimes \omega^a)$$

*and*

$$E_2^{2a+1,b} = H^b(\mathcal{M}_{FG}; \mathcal{F}_X^{\text{odd}} \otimes \omega^a)$$

## 10.2 The stratification on $\mathcal{M}_{FG}$

Recall from theorem 9.1.4 that we have a rational isomorphism $L \to \mathbb{Z}[b_1, \dots]$. This implies that over $\mathbb{Q}$, all formal group laws are uniquely isomorphic to the additive formal group law. This implies the following.

**Corollary 10.2.1.** *$\mathcal{M}_{FG}^s \times \operatorname{Spec}\mathbb{Q} \simeq \operatorname{Spec}\mathbb{Q}$, and $\mathcal{M}_{FG} \times \operatorname{Spec}\mathbb{Q} \simeq B\mathbb{G}_m \times \mathbb{Q}$. Here $B\mathbb{G}_m$ is the classifying stack of a group scheme, as usual.*

**Example 1.** *Let $f(x, y) = x+y+xy$, the multiplicative formal group law. Then the isomorphism between $f$ and the additive formal group law is given by $t \mapsto e^t - 1$, which has rational coefficients. This implies that the two formal group laws are not generally isomorphic. We would want an invariant to tell the two formal group laws apart. To do this, we make the following definition.*

**Definition 10.2.2.** For $f(x, y)$ a formal group law over a ring $R$, we define its $n$-series $[n](t) \in R[[t]]$ as follows.

1. $[0](t) = 0$.
2. $[n](t) = f([n-1](t), t)$.

**Definition 10.2.3.** We define a homomorphism $h$ between formal group laws $f$ and $f'$ to be a power series in $tR[[t]]$ such that $f(h(x), h(y)) = h(f'(x, y))$.

**Proposition 10.2.4.** *$[n]$ is a homomorphism from $f$ to itself, that is, $[n]f(x, y) = f([n]x, [n]y)$.*

*Proof.* We prove by induction. The case $n = 0$ is obvious. Suppose the proposition is correct for $n-1$, then we have the following equations.

$$\begin{aligned}
&[n]f(x,y) \\
=&f(f(x,y), [n-1]f(x,y)) \\
=&f(f(x,y), f([n-1]x, [n-1]y)) \\
=&f(x, f(y, f([n-1]y, [n-1]x))) \qquad \text{by associativity} \\
=&f(x, f(f(y, [n-1]y), [n-1]x)) \\
=&f(x, f([n]y, [n-1]x)) \\
=&f(f(x, [n-1]x), [n]y) \\
=&f([n]x, [n]y)
\end{aligned}$$

□

We wish to develop other properties of $[n]$. Therefore we introduce the following definition.

**Definition 10.2.5.** Let $R[[t]]\mathrm{d}t$ denote a free module of rank 1 over $R[[t]]$. For a formal group law $f(x, y)$, we denote

$$f^*(g(t)\mathrm{d}t) = g(f(x,y))\left(\frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y\right) \in R[[x,y]]\{\mathrm{d}x, \mathrm{d}y\}$$

We define $g(t)\mathrm{d}t$ to be a translation invariant differential if $f^*(g(t)\mathrm{d}t) = g(x)\mathrm{d}x + g(y)\mathrm{d}y$.

**Example 2.** $\mathrm{d}t$ *is a translation invariant differential for* $f(x, y) = x + y$.
$\mathrm{d}t/(1+t)$ *is a translation invariant differential for* $f(x, y) = x + y + xy$.

**Lemma 10.2.6.** *For any formal group law* $f$*, there is a unique translation invariant differential* $\omega_f$ *of the form* $(1 + c_1 t + \cdots)\mathrm{d}t$*. Moreover,* $R[[t]]\mathrm{d}t \simeq R[[t]]\omega_f$*.*

*Proof.* Direct. □

**Lemma 10.2.7.** *Let* $f$ *and* $f'$ *be formal group laws over* $R$*. Let* $h$ *be a homomorphism from* $f$ *to* $f'$*. Then* $h^*\omega_{f'} = \lambda\omega_f$*, where* $\lambda \in R$ *is a constant. Here*

$$h^*(g(t)\mathrm{d}t) = g(h(t))(\frac{\partial h}{\partial t}\mathrm{d}t)$$

*Proof.* Direct. □

**Corollary 10.2.8.** *For* $h$ *a homomorphism between formal group laws* $f$ *and* $f'$ *over a ring with characteristic* $p$*, one of the following holds.*

1. *There is some* $\lambda \neq 0$ *such that* $h(t) = \lambda t + \cdots$*.*
2. *There is some* $n$ *and* $\lambda \neq 0$ *such that* $h(t) = h'(t^{p^n})$*, and* $h'(t) = \lambda t + \cdots$*.*

*Proof.* If the $\lambda$ in lemma 10.2.7 is nonzero, then this fits in the first case. Otherwise, $h^*$ sends $\omega_{f'}$ to 0. However, since $\omega_{f'}$ is a generator of $R[[t]]\mathrm{d}t$, we have $h^* = 0$. Then unwinding the definitions, we have $\mathrm{d}h = 0$. Since $R$ has characteristic $p$, this implies all non-zero terms in $h$ can only have the degree divisable by $p$. Therefore $h(t) = h_1(t^p)$. Then we notice that $h_1$ is also a homomorphism between formal group laws, namely, $f_1(x, y) = f^p(x, y)$ and $f_1'(x, y) = f'(x^p, y^p)$. Explicitly, we have

$$h_1 f^p(x,y) = hf(x,y) = f'(h(x), h(y)) = f'(h_1(x^p), h_1(y^p)) = f'(h_1^p(x), h_1^p(y)) = f_1'(h_1(x), h_1(y))$$

Therefore, we have either $h_1$ having non-zero $\lambda$ or $h_1(t) = h_2(t^p)$. This process ends somewhere, say at $h_n$. Then $h_n$ is the $h'$ required. □

**Corollary 10.2.9.** *Either* $[p] = 0$ *or* $[p] = \lambda t^{p^n} + \cdots$*.*

*Proof.* $[n]$ is a homomorphism. □

**Definition 10.2.10.** For a formal group law $f$ and a fixed prime $p$, we let $v_n$ denote the coefficient of $t^{p^n}$ in the $p$-series $[p]$. We say $f$ is of height $\geq n$ if $v_i = 0$ for $i < n$, and height exactly $n$ if $f$ has height $\geq n$ and $v_n$ is invertible.

**Example 3.** $v_0 = p$*. Thus* $f$ *has height* $\geq 0$ *iff* $p = 0$ *in* $R$*, and height exactly* $0$ *iff* $p$ *is invertible in* $R$*.*
*The additive formal group law has height infinity if* $p = 0$ *in* $R$*, and the multiplicative formal group law has height* $1$*. Therefore, the two formal group laws are not isomorphic in rings with some* $p = 0$*.*

From now on, to focus on the prime $p$, we always assume the ring $R$ we are considering is a $\mathbb{Z}_{(p)}$-algebra, so that any other prime is invertible. We then consider the universal $v_n$'s associated to the universal formal group law on the Lazard ring. Recall that in the proof of theorem 9.1.4, the $t_i$'s are not canonically determined, but chosen to be generators of the canonically deterimined $(I/I^2)_{2n}$. We consider the image of $v_i$ in it.

**Proposition 10.2.11.** *The image of* $v_n \in (I/I^2)_{2(p^n-1)} \simeq \mathbb{Z}$ *is* $p^{p^n-1} - 1$*.*

*Proof.* Let $k = p^n - 1$. The homomorphism $L \to \mathbb{Z} \oplus \mathbb{Z}t_k$ we constructed in the proof of theorem 9.1.4 classifies the formal group law

$$f(x,y) = x + y + \sum_{0<i<p^n} \frac{1}{p}\binom{p^n}{i} t_k x^i y^{p^n - i} = x + y + \frac{t_k}{p}((x+y)^{p^n} - x^{p^n} - y^{p^n})$$

Then notice by induction that

$$[a] = at + \frac{t_k}{p}((at)^{p^n} - at^{p^n})$$

Thus the coefficient of $t^{p^n}$ in $[p]t$ is

$$\frac{t_k}{p}(p^{p^n} - p) = (p^{p^n - 1} - 1)t_k$$

$\square$

**Corollary 10.2.12.** *In the identification $L \otimes \mathbb{Z}_{(p)} \simeq \mathbb{Z}_{(p)}[t_1, t_2, \dots]$, we may let each $t_{p^n-1}$ be given by $v_n$. Equivalently, $v_n$ are the generators of the first degree Adams filtration of $\pi_*(MU)$.*

**Corollary 10.2.13.** *For a field $k$ of characteristic $p$, for all $1 \leq n \leq \infty$, there is a formal group law $f$ having height exactly $n$.*

*Proof.* The case $n = \infty$ is given by the additive formal group law. Otherwise, we let $f$ be the formal group law classified by $\mathbb{Z}_{(p)}[t_1, \dots]$ with $t_i \mapsto 0$ for $i < p^n - 1$, and $v_n \mapsto 1$. $\square$

**Lemma 10.2.14.** *The condition that a formal group law has height $\geq n$ or exactly $n$ is Zariski local.*

*Proof.* Consider an element $x \in R$. Notice that $x$ is zero or invertible iff its image in $R[a_i^{-1}]$ is zero or invertible for all $i$ for some $\sum a_i = 1$. $\square$

This gives rise to the following definition.

**Definition 10.2.15.** For a not necessarily coordinatizable formal group $\mathcal{F} : \mathrm{Alg}_R \to \mathbf{Ab}$ over $R$, we define it to have height $\geq n$ or exactly $n$ iff $\mathcal{F}_{\mathrm{Alg}_{R'}}$ has height $\geq n$ or exactly $n$ in the above sense for all $R'$ such that $\mathcal{F}_{\mathrm{Alg}_{R'}}$ is coordinatizable.

**Definition 10.2.16.** We define closed substacks of $\mathcal{M}_{FG} \otimes \mathbb{Z}_{(p)}$ with $\mathcal{M}_{FG}^{\geq n}$ given by $\mathrm{Spec}(L_{(p)}/(v_0, v_1, \dots, v_{n-1}))/G^+$, and $\mathcal{M}_{FG}^{n}$ given by $\mathrm{Spec}(L_p/(v_0, v_1, \dots, v_{n-1})[v_n^{-1}])/G^+$. By their constructions, we see that $\mathcal{M}_{FG}^{\geq n}$ classifies formal groups over $R$ of height $\geq n$ and $\mathcal{M}_{FG}^{n}$ classifies formal groups of height exactly $n$.

We actually have the following results (the proofs are elementary).

**Theorem 10.2.17.** *For any $R$ in which $p = 0$, and $f$ is a formal group law over $R$ having infinite height, then $f$ is isomorphic to the additive formal group law.*

**Theorem 10.2.18.** *Let $f$ and $f'$ be formal group laws of height exactly $n$ over $R$, and let $R'$ be defined as the ring parametrizing the isomorphisms between $f$ and $f'$ (explicitly, this is the ring $R[b_0^{\pm 1}, b_1, \dots]/I$, where $I$ is the ideal generated by the coefficients of $x^i y^j$ of the expression $gf(x,y) - f'(g(x), g(y))$ in which $g = b_0 t + b_1 t^2 + \cdots$). Then $R'$ is isomorphic to the direct limit of a system of injective finite etale maps*

$$R = R(0) \hookrightarrow R(1) \hookrightarrow R(2) \hookrightarrow \cdots$$

*In particular, if $R$ is strictly Henselian (for example when $R$ is an algebraically closed field of characteristic $p$), we have a map $R' \to R$, that is, $f$ and $f'$ are always isomorphic formal group laws.*

*Proof.* Omitted. $\square$

## 10.3 The Landweber exact functor theorem

Recall in previous sections, each complex oriented cohomology theory determines an algebra $\pi_*(E)$ over $L \simeq \pi_*(MU)$. We will answer in this section the converse problem, that is, construct complex oriented cohomology theory using algebras over $L$. However, due to many reasons (we will see this later), we will consider quasi-coherent sheaves over $\mathcal{M}_{FG}$ instead of $\operatorname{Spec} L$. Then, recall the definition of flat quasi-coherent sheaves over an algebraic stack from definition 14.2.5.

**Lemma 10.3.1.** *A coherent sheaf $M$ over $\mathcal{M}_{FG}$ is flat iff for every $\eta \in \mathcal{M}_{FG}(R)$ representing a coordinatizable formal group over* $\operatorname{Spec} R$*, we have $M(\eta)$ is flat.*

*Proof.* Recall the notion of flatness (or faithfully flatness) of a coherent sheaf on a scheme is Zariski local. Since $\eta$ represents a formal group law (not necessarily coordinatizable), we consider its localization at $a_i$, where $\sum a_i = 1$, where it is coordinatizable on $D(a_i)$. Hence $M(\eta)$ is flat over $\operatorname{Spec} R$ iff $M(\eta|_{D(a_i)})$ is flat over $\operatorname{Spec} R[a_i^{-1}]$, which proves the lemma. □

**Corollary 10.3.2.** *Consider the $L$-point $\eta_0$ representing the universal formal group law. Then we have $M$ is flat iff $M(\eta_0)$ is flat over $L$.*

*Proof.* For $\eta \in \mathcal{M}_{FG}(R)$ representing a coordinatizable formal group law, $M(\eta)$ is clearly an image of $M(\eta_0)$. Then by the definition of a quasi-coherent sheaf over an algebraic stack, we have $M(\eta) \simeq M(\eta_0) \otimes_L R$. Therefore $M(\eta)$ is flat over $R$ iff $M(\eta_0)$ is flat over $L$. □

**Construction 10.3.3.** For an $R$-point $\eta$ of $\mathcal{M}_{FG}$ corresponding to a morphism $q : \operatorname{Spec} R \to \mathcal{M}_{FG}$, the forgetful functor $q^* : \mathbf{QCoh}(\mathcal{M}_{FG}) \to \mathbf{QCoh}(\operatorname{Spec} R)$ given by $M \mapsto M(\eta)$ has a left adjoint $q_* : \mathbf{QCoh}(\operatorname{Spec} R) \to \mathbf{QCoh}(\mathcal{M}_{FG})$, which can be written as $q_*(N)(\eta') = N \otimes_R B$, where $\eta' \in \mathcal{M}_{FG}(R')$ classifies a morphism $q' : \operatorname{Spec} R' \to \mathcal{M}_{FG}$ and $B$ is the $R \otimes R'$ algebra determined by the pullback

$$\begin{tikzcd}
\operatorname{Spec} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \operatorname{Spec} R \arrow[d] \\
\operatorname{Spec} R' \arrow[r] & \mathcal{M}_{FG}
\end{tikzcd}$$

Notice that $B$ is actually the algebra classifying the universal isomorphism of the two formal groups over $R$ and $R'$. We will use this fact later.

**Definition 10.3.4.** We say an $R$-module $N$ is flat (faithfully flat) over $\mathcal{M}_{FG}$ iff $q_*(N)$ is flat (faithfully flat) over $\mathcal{M}_{FG}$. We say $q$ is flat (faithfully flat) if $R$ (as a module) is flat (faithfully flat) over $\mathcal{M}_{FG}$.

**Proposition 10.3.5.** *Let $q : \operatorname{Spec} R \to \mathcal{M}_{FG}$ be a map classifying the formal group $\eta \in \mathcal{M}_{FG}(R)$, and let $N$ be an $R$-module flat over $\mathcal{M}_{FG}$. Then the functor $M \mapsto M(\eta) \otimes_R N$ is an exact functor from $\mathbf{QCoh}(\mathcal{M}_{FG}) \to \mathbf{QCoh}(\operatorname{Spec} R) \simeq \mathbf{Mod}_R$.*

*Proof.* The problem is local on $\operatorname{Spec} R$ since exactness of quasi-coherent sheaves is a local property. Therefore we may assume $\eta$ classifies a coordinatizable formal group $\mathcal{G}_f$. Let $p : \operatorname{Spec} L \to \mathcal{M}_{FG}$ classifies the universal formal group law. Notice that the universal isomorphism between $f_{univ}$ and $f$ is given by the ring $R[b_0^{\pm 1}, b_1, \dots]$, with structure morphisms from $R$ the obvious map and from $L$ classifying the formal group law given by the action of $g(t) = b_0 t + b_1 t^2 + \cdots$ on the image of $f$. That is, we have a pullback diagram

$$\begin{tikzcd}
\operatorname{Spec} R[b_0^{\pm 1}, b_1, \dots] \arrow[r, "q'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\ulcorner"] & \operatorname{Spec} L \arrow[d, "p"] \\
\operatorname{Spec} R \arrow[r, "q"'] & \mathcal{M}_{FG}
\end{tikzcd}$$

Since the polynomial algebra is a free module over $R$, we have $p'$ a faithfully flat morphism. Hence, to show $M \mapsto q^*(M) \otimes_R N$ is exact, it suffices to show that $M \mapsto (p')^*(q^*(M) \otimes_R N)$ is exact. Explicitly, we have $(p')^*(q^*(M) \otimes_R N) \simeq (qp')^*(M) \otimes_{R[b_0^{\pm 1}, b_1, \dots]} (p')^*(N) \simeq p^*(M) \otimes_L (p')^*(N)$. However, flatness of $N$ over $\mathcal{M}_{FG}$ guarantees the flatness of $(p')^*N$ over $L$, which proves the theorem. □

**Corollary 10.3.6.** *Let $M$ be a graded module over $L$. If $M$ is flat over $\mathcal{M}_{FG}$, then the functor $X \mapsto MU_*(X) \otimes_L M$ is a homology theory representable by some spectrum $E$. Later, we will always assume the grading on $M$ is even (though the even grading is not required for this corollary).*

*Proof.* Directly verify the axioms of homology theories. □

**Proposition 10.3.7.** *$R = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ as an $L$-module is flat over $\mathcal{M}_{FG}$.*

*Proof.* We prove the morphism $\operatorname{Spec} R \to \operatorname{Spec} L \to \mathcal{M}_{FG}$ is exact. Again, we consider the diagram of the above form.

$$\begin{tikzcd} \operatorname{Spec} R[b_0^{\pm 1}, b_1, \dots] := \operatorname{Spec} B \arrow[r, "q'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\ulcorner"] & \operatorname{Spec} L \arrow[d, "p"] \\ \operatorname{Spec} R \arrow[r, "q"'] & \mathcal{M}_{FG} \end{tikzcd}$$

Now the left map is faithfully flat, hence the bottom map is flat iff the composition $q \circ p'$ is flat. Since $\operatorname{Spec} L \to \mathcal{M}_{FG}$ is flat, it suffices to prove $q'$ is flat. However, the map $q'$ exhibits $R[b_0^{\pm 1}, b_1, \dots]$ as the quotient of $L_{(p)}[b_0^{\pm 1}, b_1, \dots]$ with the ideal generated by the image of $t_i$ $(i \neq p^r - 1)$. Finally recall that for $i \neq p^r - 1$, the image of $t_i$ is given by $\lambda b_i + \cdots$, where $\lambda$ is invertible in $\mathbb{Z}_{(p)}$ and the rest terms are decomposable. Thus we may replace $b_i$ by the image of $t_i$. Therefore $B$ is a polynomial over $L_{(p)}[b_0^{\pm 1}]$ and hence flat over $L$. □

**Corollary 10.3.8.** $\operatorname{Spec} \mathbb{Z}_{(p)}[v_1, \dots]$ *is faithfully flat over the localization $\mathcal{M}_{FG} \times \mathbb{Z}_{(p)}$.*

**Corollary 10.3.9.** *The construction $X \mapsto MU_*(X) \otimes_L R \simeq MU_*(X)_{(p)}/(t_i, i \neq p^r - 1)$ is represented by a homology theory $BP_*(X)$. This is called the Brown-Peterson spectrum.*

Then we present an extremely useful theorem called the Landweber exact functor theorem. This gives a criterion on flatness.

**Definition 10.3.10.** Recall $x_i \in R$ is called a regular sequence for a module $M$ if for all $n$, $x_n$ is not a zero divisor on $M/(x_0, x_1, \dots, x_{n-1})M$. In particular, every sequence is a regular sequence for a trivial module.

**Theorem 10.3.11.** *For a module $M$ over $L$, $M$ is flat over $\mathcal{M}_{FG}$ iff for every prime $p$, $v_0, v_1, \dots$ is a regular sequence for $M$. We denote such an $M$ to be Landweber-exact.*

**Definition 10.3.12.** By corollary 10.3.6, we obviously have a Landweber-exact evenly graded module $M$ corresponds to a spectrum $E$. We also call the spectra arising from this construction Landweber-exact.

**Corollary 10.3.13.** *Let $R = \mathbb{Z}[\beta, \beta^{-1}]$, where $\beta$ has degree 2. Then the map $L \to R$ classifying the formal group law $x + y + \beta xy$ exhibits $R$ as a Landweber-exact ring.*

*Proof.* We fix a prime $p$. Then we obviously have $v_0 = p$ not a zero divisor. Then notice the $[p]$ series is given by

$$[p]t = \sum_{n=1}^{p} \binom{p}{n} \beta^{n-1} t^n$$

hence $v_1$ is invertible on $R/v_0R = R/pR$. Therefore $R$ is Landweber-exact. □

*Remark* 17. We will later see that this represents the complex $K$-theory.

**Construction 10.3.14.** Let $R$ be a commutative ring and $E$ be an elliptic curve defined over $R$. We can associate to $E$ a formal group $\widehat{E}$ given by $\widehat{E}(A) \subseteq \mathrm{Hom}(\operatorname{Spec} A, E)$ to be all points such that the diagram commutes

$$\begin{tikzcd} \operatorname{Spec} A/\mathfrak{n}(A) \arrow[r] \arrow[d] & \operatorname{Spec} A \arrow[d] \\ \operatorname{Spec} R \arrow[r, "0"'] & E \end{tikzcd}$$

where the left map is given by the composition $\operatorname{Spec} A/\mathfrak{n}(A) \to \operatorname{Spec} A \to E \to \operatorname{Spec} R$. Moreover, the multiplication is given by the operation on the elliptic curve. This construction gives a map $\mathcal{M}_{Ell} \to \mathcal{M}_{FG}$. Finish the construction of elliptic cohomology

We now present a proof of theorem 10.3.11. Again, flatness is a local condition, hence it suffices to prove $M_{(p)}$ is flat over $\mathcal{M}_{FG} \otimes \mathbb{Z}_{(p)}$ for all $p$. We fix a prime $p$. Before the proof, we first establish a few conclusions.

**Lemma 10.3.15.** *Recall the construction $q : \mathbb{Z}_{(p)}[v_1, \dots] \to \mathcal{M}_{FG}$ representing the $BP$ spectrum in proposition 10.3.7. Then a quasi-coherent sheaf $M$ over $\mathcal{M}_{FG}$ is flat iff $q^*(M)$ is a flat module over $\mathbb{Z}_{(p)}[v_1, \dots]$.*

*Proof.* $\Longrightarrow$ : Obvious since $\mathbb{Z}_{(p)}[v_1, \dots]$ is flat over $\mathcal{M}_{FG}$.
$\Longleftarrow$ : We fix a map $f : \operatorname{Spec} R \to \mathcal{M}_{FG}$. We wish to show $f^*(M)$ is flat. By corollary 10.3.8, we form the following pullback

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r, "q'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\ulcorner", very near start] & \operatorname{Spec} L \arrow[d, "f"] \\ \operatorname{Spec} \mathbb{Z}_{(p)}[v_1, \dots] \arrow[r, "q"'] & \mathcal{M}_{FG} \times \mathbb{Z}_{(p)} \end{tikzcd}$$

and the top morphism is faithfully flat. Therefore it suffices to prove the $(f \circ q')^*(M)$ is flat over $B$. But this is equivalent to the module $f^*(M) \otimes_R B \simeq q^*(M) \otimes_{\mathbb{Z}_{(p)}[v_1,\dots]} B$ by considering the other composition, which is flat over $B$ iff $q^*(M)$ is flat over $\mathbb{Z}_{(p)}$. $\square$

**Lemma 10.3.16.** *Let $R$ be a commutative ring containing a non-zero-divisor $x$, and let $M$ be an $R$-module. Then $M$ is flat over $M$ iff the three following conditions are satisfied.*

1. *$x$ is a non-zero-divisor on $M$.*
2. *The quotient $M/xM$ is a flat $R/(x)$-module.*
3. *The module $M[x^{-1}]$ is a flat $R[x^{-1}]$-module.*

*Proof.* $\Longrightarrow$ : Obvious.
$\Longleftarrow$ : We prove $\mathrm{Tor}_i^R(M, N) \simeq 0$ for all $N$.
We first prove the case where $N$ consists of $x$-power-torsions. We may write $N$ as a filtered colimit $N_i$, where $N_i$ consists of the elements annihilated by $x^i$. Then, consider the exact sequence

$$0 \longrightarrow K \longrightarrow N_i \longrightarrow xN_i \longrightarrow 0$$

Now, $N_i$ is $x^{i-1}$-torsion and $K$ is $x$ torsion, and by the long exact sequence of Tor, it suffices to prove for $N$ with $xN = 0$. These $N$'s are $R/(x)$ modules, hence by the universal coefficient theorem we have $\mathrm{Tor}_i^R(M, N) \simeq \mathrm{Tor}_i^{R/(x)}(M/xM, N) = 0$, by the second assumption.

Then, we prove for general $N$. We consider $K = \ker(N \to N[x^{-1}])$. Then $K$ is obviously $x$-power-torsion, hence to prove $\mathrm{Tor}_i^R(M,N) = 0$ it suffices to prove $\mathrm{Tor}_i^R(M, N/K) = 0$. That is, we may assume $N \to N[x^{-1}]$ is injective. Then we may consider the exact sequence

$$0 \longrightarrow N \longrightarrow N[x^{-1}] \longrightarrow C \longrightarrow 0$$

Since $C$ is $x$-power-torsion, it suffices to show $\mathrm{Tor}_i^R(M, N[x^{-1}]) = 0$. Finally, in this case, $N \simeq N[x^{-1}]$ is a module over $R[x^{-1}]$. Then again by the third assumption and the universal coefficient theorem $\mathrm{Tor}_i^R(M,N) \simeq \mathrm{Tor}_i^{R[x^{-1}]}(M[x^{-1}], N) = 0$. $\square$

**Lemma 10.3.17.** *Every quasi-coherent sheaf on the stack $\mathcal{M}_{FG}^m$ is flat (recall from definition 10.2.16 the definition of $\mathcal{M}_{FG}^m$).*

*Proof.* For any map $q : \operatorname{Spec} A \to \mathcal{M}_{FG}^m$ classifying a formal group of height exactly $m$ on $A$, and $X$ a quasi-coherent sheaf on $\mathcal{M}_{FG}$, we prove $q^*$ is flat over $A$. Since again flatness is a local notion, we may suppose the formal group classified by $q$ is coordinatizable. Then, choose a formal group law of height $m$ over $\mathbb{F}_p$, classified by a map $f : \operatorname{Spec} \mathbb{F}_p \to \mathcal{M}_{FG}^m$. We form the pullback

$$\begin{tikzcd} \operatorname{Spec} A' \arrow[r] \arrow[d] & \operatorname{Spec} \mathbb{F}_p \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r, "q"'] & \mathcal{M}_{FG} \end{tikzcd}$$

In fact, by theorem 10.2.18, $A'$ is a direct limit of injective finite etale ring extensions. Therefore $A'$ is faithfully flat over $A$. Therefore it suffices to prove $q^*(X) \otimes_A A'$ is flat over $A'$, and $q^*(X) \otimes_A A' \simeq f^*(X) \otimes_{\mathbb{F}_p} A'$, which is flat over $\mathbb{F}_p$, hence flat over $A'$. $\square$

*Proof.* We prove theorem 10.3.11. We first prove $v_i$ being a regular sequence implies flantess. Now, let $M$ be a module over $L_{(p)}$. Since for $p' \neq p$, $v_0 = p'$ is already invertible hence $v_i$ is a regular sequence, it suffices to consider the case $p' = p$. We wish to prove for $M$ such that $v_0 = p, v_1, \dots$ is a regular sequence, the pushforward of $M$ along $\operatorname{Spec} L_{(p)} \to \mathcal{M}_{FG} \times \operatorname{Spec} \mathbb{Z}_{(p)}$ is flat. We form the pullback square

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \operatorname{Spec} L_{(p)} \arrow[d] \\ \operatorname{Spec} \mathbb{Z}_{(p)}[v_1, \dots] \arrow[r] & \mathcal{M}_{FG} \times \operatorname{Spec} \mathbb{Z}_{(p)} \end{tikzcd}$$

By lemma 10.3.15, it suffices to prove $M_B := M \otimes_{L_{(p)}} B$ is a flat module over the ring $\mathbb{Z}_{(p)}[v_1, \dots]$. We attempt to prove $\mathrm{Tor}_i^{\mathbb{Z}_{(p)}[v_1,\dots]}(M_B, N)$ vanish.
Recall the functor Tor commutes with filtered colimits, hence it suffices to show the case where $N$ is finitely presented. The finite presentation gives $N \simeq N_0[v_{n+1}, v_{n+2}, \dots]$, where $N_0$ is a module over $\mathbb{Z}_{(p)}[v_1, \dots, v_n]$. By the universal coefficient theorem we have $\mathrm{Tor}_i^{\mathbb{Z}_{(p)}[v_1,\dots]}(M_B, N) \simeq \mathrm{Tor}_i^{\mathbb{Z}_{(p)}[v_1,\dots,v_n]}(M_B, N_0)$. Therefore it suffices to prove $M_B$ is flat over all $\mathbb{Z}_{(p)}$.
Notice that the the homomorphisms $\mathbb{Z}_{(p)}[v_1, \dots] \to B$ and $L_{(p)} \to B$ induces different images of $v_i$ in $B$, say $v_i'$ and $v_i''$. However we still have the isomorphism $(v_1', \dots, v_m') \simeq (v_1'', \dots, v_m'')$. We denote this ideal by $I_{m+1}$, then prove inductively $M_B/I_m M_B$ is flat over
$R := \mathbb{Z}_{(p)}[v_1, \dots, v_n]/(p, v_1, \dots, v_{m-1})$ for $m \leq n+1$. Notice when $m = n+1$, the ring is isomorphic to $\mathbb{F}_p$, hence is obvious. Then notice $M_B/I_m M_B \simeq B \otimes_{L_{(p)}} (M/(v_0, \dots, v_{m-1}))$. Since $v_m$ is not a zero divisor on $M/(v_0, \dots, v_{m-1})$ by assumption, and since $B$ is flat over $L_{(p)}$, we have an exact sequence

$$0 \longrightarrow M_B/I_m M_B \xrightarrow{v_m''} M_B/I_m M_B \longrightarrow M_B/I_{m+1} M_B$$

Since $v''_m \equiv \lambda v'_m (\mod I_m)$ where $\lambda$ is invertible, $v_m \in R$ is a non-zero-divisor on $M_B/I_m M_B$. Moreover, the quotient $M_B/(I_m, v_m)M_B \simeq M_B/I_{m+1}M_B$ is flat over $R/(v_m)$ by the induction hypothesis. By lemma 10.3.16, it suffices to prove $(M_B/I_m M_B)[v_m^{-1}]$ is flat over $R[v_m]^{-1}$, which follows from lemma 10.3.17. The converse follows exactly from reversing the proof. $\square$

## 10.4 Morava stablizer groups

**Construction 10.4.1.** Recall from theorem 10.2.18 we have a unique formal group law over $\overline{\mathbb{F}}_p$ of height $n$, say $f$. We consider the group $G = \operatorname{Aut}(\operatorname{Spf} \overline{\mathbb{F}}_p[[t]])$, where the automorphism is required to respect the formal group structure and not required to be compatible with the structure map $\operatorname{Spf} \overline{\mathbb{F}}_p[[t]] \to \operatorname{Spec} \overline{\mathbb{F}}_p$. Obviously we have an exact sequence

$$0 \longrightarrow \operatorname{Aut}_{\overline{\mathbb{F}}_p}(\operatorname{Spf} \overline{\mathbb{F}}_p[[t]]) \longrightarrow \operatorname{Aut}(\operatorname{Spf} \overline{\mathbb{F}}_p[[t]]) \longrightarrow \operatorname{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \longrightarrow 0$$

where $\operatorname{Aut}_{\overline{\mathbb{F}}_p}(\operatorname{Spf} \overline{\mathbb{F}}_p[[t]])$ is the subgroup of $\operatorname{Aut}(\operatorname{Spf} \overline{\mathbb{F}}_p[[t]])$ respecting the structure map.

**Theorem 10.4.2.** *We have $\mathcal{M}^n_{FG} \simeq \operatorname{Spec} \overline{\mathbb{F}}_p/G$, where $G$ acts on $\overline{\mathbb{F}}_p$ by the projection $G \to \operatorname{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$.*

Before considering $G$, we first consider its subgroup $\operatorname{Aut}_{\overline{\mathbb{F}}_p}(\operatorname{Spf} \overline{\mathbb{F}}_p[[t]])$. We recall that $\operatorname{Aut}_{\overline{\mathbb{F}}_p}(\operatorname{Spf} \overline{\mathbb{F}}_p[[t]])$ is the group of units of the ring $\operatorname{End}_{\overline{\mathbb{F}}_p}(\operatorname{Spf} \overline{\mathbb{F}}_p[[t]])$, whose elements are power series $g(t) \in \overline{\mathbb{F}}_p[[t]]$ satisfying $g(f(x,y)) = f(g(x), g(y))$.

**Construction 10.4.3.** Let $f^p$ denote the formal group law over $\overline{\mathbb{F}}_p$ given by applying the Frobenius map to the coefficients of $f$. Then $f^p$ is still a formal group law of height $n$, hence non-canonically isomorphic to $f$ by theorem 10.2.18, say $\nu$. Since $f(x,y)^p \simeq f^p(x^p, y^p)$, we have

$$\nu(f(x,y)^p) \simeq \nu(f^p(x^p, y^p)) \simeq f(\nu(x^p), \nu(y^p))$$

Thus $\pi(t) = \nu(t^p)$ is an endomorphism of $f$.
Then, let $g \in \operatorname{End}_{\overline{\mathbb{F}}_p}(\operatorname{Spf} \overline{\mathbb{F}}_p[[t]])$ be arbitrary, and let $g(t) = b_0 t + b_1 t^2 + \cdots$. If $b_0 \neq 0$, then $g \in \operatorname{Aut}_{\overline{\mathbb{F}}_p}(\operatorname{Spf} \overline{\mathbb{F}}_p[[t]])$. Otherwise $g(t) = g_0(t^p)$ by corollary 10.2.8, and $g_0$ is an endomorphism of $f^p$. Thus $g_0 \circ \nu^{-1}$ is an endomorphism of $f$, and we have $g = g_0 \circ \nu^{-1} \circ \nu \circ \operatorname{Frob}_p = (g_0 \circ \nu^{-1}) \circ \pi$. Thus, every non-invertible element $g$ of the ring $\operatorname{End}_{\overline{\mathbb{F}}_p}(\operatorname{Spf} \overline{\mathbb{F}}_p[[t]])$ can be written uniquely of the form $g'\pi$.
Therefore, the ring $\operatorname{End}_{\overline{\mathbb{F}}_p}(\operatorname{Spf} \overline{\mathbb{F}}_p[[t]])$ is a (non-commutative) local ring with the non-invertible elements a two-sided ideal. This ideal is the left ideal generated by $\pi$. Moreover, by corollary 10.2.8, we see that every endomorphism $g$ is of the form $u\pi^k$ for some $k$. We will refer to $k$ as the valuation of $g$, and denoted by $v(g)$. By convention we set $v(0) = \infty$. Notice we then have $v([p]) = n$, where $n$ is the height of $f$. Also, although the ring is non-commutative, we still have $v(gg') = v(g) + v(g')$.

**Construction 10.4.4.** We let $\lambda : \operatorname{End}_{\overline{\mathbb{F}}_p}(\operatorname{Spf} \overline{\mathbb{F}}_p[[t]]) \to \overline{\mathbb{F}}_p$ sending $g(t) = b_0 t + b_1 t^2 + \cdots$ to $b_0$. Then, consider $[p](t) = \mu t^{p^n} + \cdots$. Since $g$ is a homomorphism of $f$, we have $[p](g(t)) = g([p](t))$, which implies $b_0 \mu t^{p^n} + \cdots = \mu b_0^{p^n} t^{p^n}$, hence $b_0 \in \mathbb{F}_{p^n}$. Thus $\lambda : \operatorname{End}_{\overline{\mathbb{F}}_p}(\operatorname{Spf} \overline{\mathbb{F}}_p[[t]]) \to \mathbb{F}_{p^n}$ is surjective.

**Construction 10.4.5.** Obviously we have $\operatorname{End}_{\overline{\mathbb{F}}_p}(\operatorname{Spf} \overline{\mathbb{F}}_p[[t]]) \simeq \varprojlim(\operatorname{End}_{\overline{\mathbb{F}}_p}(\operatorname{Spf} \overline{\mathbb{F}}_p[[t]])/(\pi^k))$. Then, notice that $\operatorname{End}_{\overline{\mathbb{F}}_p}(\operatorname{Spf} \overline{\mathbb{F}}_p[[t]])/(\pi^k)$ is finite of cardinality $p^{nk}$ (notice that this is true for $k = 1$, and the rest follows from the factorization $g = u\pi^k$). Then, notice $[p^k] \in \pi^k$. This induces a well-defined map $\mathbb{Z}/(p^k) \to \operatorname{End}_{\overline{\mathbb{F}}_p}(\operatorname{Spf} \overline{\mathbb{F}}_p[[t]])/(\pi^k)$, which is compatible with the quotient maps. This induces a map $\mathbb{Z}_p \to \operatorname{End}_{\overline{\mathbb{F}}_p}(\operatorname{Spf} \overline{\mathbb{F}}_p[[t]])$, which exhibits a $\mathbb{Z}_p$ algebra structure.

Then, we consider the non-commutative ring $D = \mathrm{End}_{\overline{\mathbb{F}}_p}(\mathrm{Spf}\,\overline{\mathbb{F}}_p[[t]])[p^{-1}]$. Notice that $[p]$ is a non-zero-divisor in $\mathrm{End}_{\overline{\mathbb{F}}_p}(\mathrm{Spf}\,\overline{\mathbb{F}}_p[[t]])$, hence it can be viewed as a subring of $D$. Moreover, since $u\pi^n = p$, $\pi$ is invertible in $D$. The valuation $v$ extends to $D$ by $v(\lambda/p^k) = v(\lambda) - nk$.

**Lemma 10.4.6.** $\mathrm{End}_{\overline{\mathbb{F}}_p}(\mathrm{Spf}\,\overline{\mathbb{F}}_p[[t]]) \simeq \{x \in D | v(x) \geq 0\}$.

*Proof.* This is obvious. □

**Lemma 10.4.7.** $\mathrm{End}_{\overline{\mathbb{F}}_p}(\mathrm{Spf}\,\overline{\mathbb{F}}_p[[t]])$ *is* $n^2$ *dimensional as a vector space over* $\mathbb{Q}_p$.

*Proof.* Let $\{x_i\}$ denote the set of $\mathbb{F}_p$ basis of $\mathbb{F}_{p^n}$. Then let $g_i$ denote an element in $\mathrm{End}_{\overline{\mathbb{F}}_p}(\mathrm{Spf}\,\overline{\mathbb{F}}_p[[t]])$ such that $\lambda(g_i) = x_i$. Then clearly $\{\pi^j g_i\}$ is a basis. □

**Lemma 10.4.8.** *Let* $g \in D$. *Then the conjugate action of* $g$ *on* $\mathrm{End}_{\overline{\mathbb{F}}_p}(\mathrm{Spf}\,\overline{\mathbb{F}}_p[[t]])/(\pi) \simeq \mathbb{F}_{p^n}$ *is given by* $b \mapsto b^{v(g)}$.

*Proof.* WLOG $g \in \mathrm{End}_{\overline{\mathbb{F}}_p}(\mathrm{Spf}\,\overline{\mathbb{F}}_p[[t]])$. Then suppose $g(t) = \lambda t^{p^{v(g)}} + \cdots$. Fix $b \in \mathbb{F}_{p^n}$, and let $h(t) = bt + \cdots$. Then let $h'(t) = g \circ h \circ g^{-1}(t)$, from $g \circ h = h' \circ g^{-1}$ we have $\lambda b^{p^{v(g)}} = b'\lambda$. This gives the result. □

**Lemma 10.4.9.** *$D$ has center* $\mathbb{Q}_p$.

*Proof.* Let $g$ be in the center of $D$. We may assume $g \in \mathrm{End}_{\overline{\mathbb{F}}_p}(\mathrm{Spf}\,\overline{\mathbb{F}}_p[[t]])$, and prove $g$ is in $\mathbb{Z}_p$. Since $\mathbb{Z}_p$ is a closed subset, it suffices to show there is an integer $m$ such that $g \equiv m ( \bmod p^k)$ for all $k$. We prove by inductin. Since $\pi g \pi^{-1} = g$, the reduction of $g$ mod $\pi$ belongs to $\mathbb{F}_p \subseteq \mathbb{F}_{p^n}$. Therefore we may subtract this integer, and assume $v(g) > 0$. Then by the preceding lemma, the conjugate action of $g$ on $\pi$ is trivial, hence $v(g)$ must be divisible by $n$. Thus $g = g'p$ for some $g'$ in the center of $\mathrm{End}_{\overline{\mathbb{F}}_p}(\mathrm{Spf}\,\overline{\mathbb{F}}_p[[t]])$, and thus $g'$ is congruent to an integer mod $p^{k-1}$, which is by the induction hypothesis. □

**Construction 10.4.10.** By the preceding lemma, $D$ is a central simple algebra over $\mathbb{Q}_p$, hence can be indentified with an element of $\mathrm{Br}(\mathbb{Q}_p)$. We construct an isomorphism $\mathrm{Br}(\mathbb{Q}_p) \to \mathbb{Q}/\mathbb{Z}$ as follows. For a Brauer class, we represent it with a division algebra $D'$. Then it contains a ring of integers $\mathcal{O}$ and its maximal ideal $\mathfrak{m}$. We have a valuation $v : D' - \{0\} \to \mathbb{Z}$ with $\mathcal{O} = v^{-1}(\mathbb{Z}_{\geq 0})$ and $\mathfrak{m} = v^{-1}(\mathbb{Z}_{>0})$. Finally, the conjugation action gives a surjective homomorphism $D' - \{0\} \to \mathrm{Gal}((\mathcal{O}/\mathfrak{m})/\mathbb{F}_p)$. In particular, the frobenius automorphism on $\mathcal{O}/\mathfrak{m}$ is given by conjugation by some $x \in D'$. We define $\mu(D') = v(x)/v(p)$. Hence, $D$ is the unique central division algebra over $\mathbb{Q}_p$ with $\mu(D) = 1/n$.

**Construction 10.4.11.** Recall that we have $\mathrm{End}_{\overline{\mathbb{F}}_p}(\mathrm{Spf}\,\overline{\mathbb{F}}_p[[t]])^\times = \mathrm{Aut}_{\overline{\mathbb{F}}_p}(\mathrm{Spf}\,\overline{\mathbb{F}}_p[[t]])$. We attempt to extend this isomorphism to a map $\chi : D^\times \to \mathrm{Aut}(\mathrm{Spf}\,\overline{\mathbb{F}}_p[[t]])$. In fact, $\chi$ is given on $\mathrm{End}_{\overline{\mathbb{F}}_p}(\mathrm{Spf}\,\overline{\mathbb{F}}_p[[t]]) - \{0\}$ by sending $g$ to the map $g_0 \circ \mathrm{Frob}_p^{v(g)}$ where $g_0$ is the isomorphism of $f$ with $f^{p^{v(g)}}$ determined by $g$ and $\nu$. That is, we have a commutative diagram of short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{End}_{\overline{\mathbb{F}}_p}(\mathrm{Spf}\,\overline{\mathbb{F}}_p[[t]])^\times \arrow[r] \arrow[d] & D^\times \arrow[r, "v"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Aut}_{\overline{\mathbb{F}}_p}(\mathrm{Spf}\,\overline{\mathbb{F}}_p[[t]])^\times \arrow[r] & \mathrm{Aut}(\mathrm{Spf}\,\overline{\mathbb{F}}_p[[t]])^\times \arrow[r] & \mathrm{Gal}(\overline{\mathbb{F}}_p, \mathbb{F}_p) \arrow[r] & 0
\end{tikzcd}$$

The left map is an isomorphism by definition, and the right morphism is the profinite completion $\mathbb{Z} \to \widehat{\mathbb{Z}}$. Therefore, the middle map is somewhat of a completion.

**Proposition 10.4.12.** *The formal group law of height $n$ over $\overline{\mathbb{F}}_p$ is in the image of the push-forward map along the inclusion $\mathbb{F}_{p^n} \hookrightarrow \overline{\mathbb{F}}_p$.*

**Construction 10.4.13.** Recall $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^k})$ can be identified with $k\widehat{Z}$ with respect to the inclusion $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^k}) \hookrightarrow \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \simeq \widehat{\mathbb{Z}}$.
By descent theory, descending the formal group is equivalent to giving a $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^k}) \simeq k\widehat{\mathbb{Z}}$ action on $\mathrm{Spf}\,\overline{\mathbb{F}}_p[[t]]$, which is equivariant w.r.t the projection $\mathrm{Spf}\,\overline{\mathbb{F}}_p[[t]] \to \mathrm{Spec}\,\overline{\mathbb{F}}_p$. Also, $k\widehat{\mathbb{Z}}$ is topologically cyclic, hence this condition is equivalent to choosing a single element of $\mathrm{Aut}(\mathrm{Spf}\,\overline{\mathbb{F}}_p[[t]])$ lying over the integer $k$, that is, giving an element of $D^\times$ with $v(x) = k$. Such element exists for all $k \geq 1$, and in particular, when $k = n$, we have a canonical choice $[p]$. Unwinding the definitions, we have proven the proposition.

**Corollary 10.4.14.** *$\mathcal{M}_{FG}^n$ is equivalently $\mathbb{F}_{p^n}/G'$, where $G'$ fits in an exact sequence*

$$0 \longrightarrow \mathrm{Aut}_{\mathbb{F}_{p^n}}(\mathrm{Spf}\,\mathbb{F}_{p^n}[[t]]) \longrightarrow G' \longrightarrow \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \longrightarrow 0$$

*The group $G'$ is sometimes called also the morava stabilizer group.*

## 10.5 Lubin-Tate theory

After we have finished analysing the structure of $\mathcal{M}_{FG}^n$, we attempt to put these strata together. Therefore, we will have to figure out what $\mathcal{M}_{FG} \times \mathbb{Z}_{(p)}$ looks like near one strata, which is where deformation theory gets involved. In this section, we fix a perfect field of characteristic $p$, say $k$, and fix a formal group law of height $n$ over $k$, say $f$. We first recall that the infinitesimal objects in **Sch** (or **Aff**) are the Artinian local rings.

**Definition 10.5.1.** We define an infinitesimal thickening of a field $k$ to be an Artinian local ring $A$ whose residue field is $k$. Intuitively, $\mathrm{Spec}\,A$ is $\mathrm{Spec}\,k$ with a tiny, pointless neighborhood.

**Definition 10.5.2.** Let $A$ be an infinitesimal thickening of $k$. For a formal group law $f$ over $k$, we define a deformation of $f$ over $A$ to be a formal group law $f_A$ over $A$, and for $\varphi : A \to k$ the quotient map, $\varphi_*(f_A) = f$. We define two deformations of $f$, say $f_A$ and $f'_A$, to be isomorphic if there is an invertible power series $g(t) \in A[[t]]$ such that $gf_A(x, y) = f'_A(g(x), g(y))$, and $g \equiv t \mod \mathfrak{m}_A$. We denote the groupoid of deformations of $f$ with isomorphisms by $\mathrm{Def}_A(f)$.

**Lemma 10.5.3.** *The groupoid $\mathrm{Def}_A(f)$ is discrete.*

*Proof.* To prove this, it suffices to prove that the loopspace at each point is trivial, that is, let $f_A$ be a deformation of $f$, then $g$ an automorphism of $f_A$ such that $g \equiv t \mod \mathfrak{m}_A$ is the identity. We prove by induction that $g \equiv t \mod \mathfrak{m}_A^a$. Since $A$ is Artinian, for sufficiently large $a$ $\mathfrak{m}_A^a = 0$, and would imply the lemma. The case $a = 1$ is by assumption.
Then, let $A'$ be the quotient of $A[b_0^{\pm 1}, b_1, \dots]$ classifying automorphisms of $f_A$. Then the map $g$ is classified by a map $\psi : A' \to A$, and the identity automorphism is classified by $\psi_0 : A' \to A$. Assume that $g \equiv t \mod \mathfrak{m}_A^a$. Then the composite maps $\psi, \psi_0 : A' \to A \to A/\mathfrak{m}_A^a$ agree. Thus $d = \psi - \psi_0 : A' \to \mathfrak{m}_A^a \to V = \mathfrak{m}_A^a/\mathfrak{m}_A^{a+1}$ is a well-defined $A$-derivation. This map factors as a composition $A' \to A' \otimes_A k \xrightarrow{d'} V$, where the second map, $d'$, is a $k$-derivation. However notice that we have $A' \otimes_A k$ the ring classifying the automorphisms of $f$. Finally, by theorem 10.2.18, we have $A' \otimes_A k$ etale over $k$, hence $d' = 0$. This implies $g \equiv t \mod \mathfrak{m}_A^{a+1}$. □

**Definition 10.5.4.** For a ring $A$, we define the Witt vectors over $A$ to be infinite sequences $a = (a_0, a_1, \dots) \in A^{\mathbb{N}}$. This set admits a ring structure, with addition given by

$$(a_0, a_1, \dots) + (b_0, b_1, \dots) = (S_0(a_0, b_0), S_1(a_0, b_0, a_1, b_1), \dots)$$

and multiplication

$$(a_0, a_1, \dots)(b_0, b_1, \dots) = (M_0(a_0, b_0), M_1(a_0, b_0, a_1, b_1), \dots)$$

where $S_n$ and $M_n$ are polynomials of $x_0, \dots, x_n, y_0, \dots, y_n$ with integer coefficients determined by

$$\Phi_n(S_0, \dots, S_n) = \Phi_n(x_0, \dots, x_n) + \Phi_n(y_0, \dots, y_n)$$

and

$$\Phi_n(M_0, \dots, M_n) = \Phi_n(x_0, \dots, x_n)\Phi_n(y_0, \dots, y_n)$$

and

$$\Phi_n(z_0, \dots, z_n) = z_0^{p^n} + pz_1^{p^{n-1}} + \dots + p^n z_n$$

This ring is denoted by $W(A)$. Notice the zero element is $(0, 0, \dots)$ and the identity is $(1, 0, \dots)$.

**Construction 10.5.5.** Let $R = W(k)[[v_1, \dots, v_{n-1}]]$. Then we have a canonical map $R \to k$ with kernel the maximal ideal $\mathfrak{m}_R = (p, v_1, \dots, v_{n-1})$. The formal group law $f$ over $k$ is classified by a map $\varphi_0 : L_{(p)} \to k$. Since $f$ has height $n$, we obviously have $t_{p^i-1}$ is send to 0 by $\varphi_0$ for $i < n$. We let $\varphi : L_{(p)} \to R$ be any map that lifts $\varphi_0$ and sends $t_{p^i-1}$ to $v_i$. Then this map determines a formal group law over $R$, denoted by $\overline{f}$. whose image is $f$ under the canonical map $R \to k$.

**Theorem 10.5.6.** *The formal group law $\overline{f}$ over $R$ is a universal deformation of $f$. That is, for any infinitesimal thickening of $k$, $\overline{f}$ induces a bijection*

$$\mathrm{Hom}_k(R, A) \xrightarrow{\simeq} \mathrm{Def}_A(f)$$

**Lemma 10.5.7.** *If $A \to A'$ is a surjective map between infinitesimal thickenings of $k$, then the induced map $\mathrm{Def}_A(f) \to \mathrm{Def}_{A'}(f)$ is surjective.*

*Proof.* This is since the Lazard ring is a polynomial ring, and we may simply lift image of the generators. ☐

**Lemma 10.5.8.** *Given a pair of surjective maps $A \to B \leftarrow C$ between infinitesimal thickenings of $k$, the canonical map $\mathrm{Def}_{A\times_B C}(f) \to \mathrm{Def}_A(f) \times_{\mathrm{Def}_B(f)} \mathrm{Def}_F(f)$ is a bijection.*

*Proof.* Notice that since Artinian local rings has only trivial Zariski open covers, a formal group over $A$ is equivalent to a formal group law over $A$. Then this conclusion is equivalent to the fact that formal groups satisfies descent. ☐

**Lemma 10.5.9.** *theorem 10.5.6 holds for $A = k[x]/(x^2)$.*

*Proof.* We first recall that $\mathrm{Hom}_k(W(k), k[x]/(x^2)) \simeq 0$ by commutative algebra, hence $\mathrm{Hom}_k(R, k[x]/(x^2)) \simeq k^{n-1}$. Notice an element in $\mathrm{Def}_A(f)$ is equivalently a map $L \to A$ with compatibility conditions. In this case, the compatibility conditions is simply $v_i \mapsto c_i x$. We construct a map $\theta' : \mathrm{Def}(k[x]/(x^2)) \to k^{n-1}$ given by $f : L \to k[x]/x^2$ mapping to $(c_1, \dots, c_{i-1})$. Thus the composition

$$\mathrm{Hom}_k(R, k[x]/(x^2)) \to \mathrm{Def}_A(f) \to k^{n-1}$$

is an isomorphism by construction. Then it suffices to prove the second map is an isomorphism. We have seen it is surjective, and we prove injectivity. But this is by lemma 10.5.3. ☐

*Proof.* We now prove theorem 10.5.6. We proceed by induction on the length of the Artinian ring $A$. If $A$ has length 1, then $A = k$ and the conclusion obviously holds. If $A$ has length $> 1$, we pick $x$ annihilated by $\mathfrak{m}_A$. By lemma 10.5.8, we have the pullback diagram

$$\begin{tikzcd}
\mathrm{Def}_{A \times_{A/(x)} A}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Def}_A(f) \arrow[d] \\
\mathrm{Def}_A(f) \arrow[r] & \mathrm{Def}_{A/(x)}(f)
\end{tikzcd}$$

Notice $A \times_{A/(x)} A \simeq k[x]/(x^2) \times_k A$. The addition map $k[x]/(x^2) \times_k k[x]/(x^2) \to k[x]/(x^2)$ determines a group structure on $\mathrm{Def}_{k[x]/(x^2)}(f)$ by lemma 10.5.8. And the multiplication $k[x]/(x^2) \times_k A \to A$ determines an action of $\mathrm{Def}_{k[x]/(x^2)}(f)$ on $\mathrm{Def}(A)$, and the pullback diagram shows that we have an embedding $\mathrm{Def}_A(f)/\mathrm{Def}_{k[x]/(x^2)}(f) \hookrightarrow \mathrm{Def}_{A/(x)}(f)$. By lemma 10.5.7 this map is also surjective. Similarly, we may show that $\mathrm{Hom}_k(R, A)/\mathrm{Hom}_k(R, k[x]/(x^2)) \simeq \mathrm{Hom}_k(R, A/(x))$. Now by lemma 10.5.9 and the induction hypothesis for $\mathrm{Hom}_k(R, A/(x))$ we obtain the result. □

**Corollary 10.5.10.** *Let $A$ be a complete Noetherian local ring with residue field $k$ and maximal ideal $\mathfrak{m}_A$. Then theorem 10.5.6 also holds for $A$.*

*Proof.* Directly write $A \simeq \varprojlim A/\mathfrak{m}_A^a$, where each term is a Artinian local ring. □

**Construction 10.5.11.** By the preceding corollary, $W(k)[[v_1, \ldots, v_{n-1}]]$ is Yonedianly determined by the residue field $k$ together with a choice of a formal group law of height $n$ over $k$. Taking $k = \overline{\mathbb{F}}_p$, we have a action of $G$ on $W(k)[[v_1, \ldots, v_{n-1}]]$, where $G$ is the Morava stabilizer group.

# Chapter 11

# Morava $E$ theory and Morava $K$ theory

## 11.1 Landweber-exact spectra

**Definition 11.1.1.** Let $f : E \to E'$ be a map of spectra. We say $f$ is a phantom if the underlying map $f_* : E_* \to E'_*$ between homology theories is the zero map.

**Proposition 11.1.2.** *Let $f : E \to E'$ be a map of spectra. The following conditions are equivalent.*

1. *The map $f$ is a phantom.*
2. *For every spectrum $X$, the map $E_*(X) \to E'_*(X)$ is zero.*
3. *For every finite spectrum $X$, the map $E_*(X) \to E'_*(X)$ is zero.*
4. *For every finite spectrum $X$, the map $E^*(X) \to E'^*(X)$ is zero.*
5. *For every finite spectrum $X$ and every map $g : X \to E$, the composition $fg : X \to E'$ is zero.*

*Proof.* (1) $\iff$ (2) $\iff$ (3) is by the fact that every spectra can be written as a filtered colimit of $\Sigma^n S$. Then (3) $\iff$ (4) by Spanier-Whitehead duality, and (4) $\iff$ (5) is obvious. $\square$

**Definition 11.1.3.** We say a finite spectrum $X$ is even if the homology groups $H\mathbb{Z}_*(X)$ are free abelian groups concentrated on even degrees. Equivalently, a finite spectrum $X$ is even if it admits a finite cell decomposition using only even dimensional cells.
We say a spectrum $E$ is evenly generated if for every map $X \to E$ where $X$ is finite, we have a factorization of the map $X \to X' \to E$, where $X'$ is even.

**Proposition 11.1.4.** *Any Landweber-exact spectrum $E$ is evenly generated.*

*Proof.* Let $E$ be the Landweber-exact spectrum associated to a graded $L$-module $M$. Then let $f : X \to E$ be a map where $X$ is finite. We may associate $f$ to an element in $E^0(X) \simeq E_0(\mathbb{D}X) = (MU \otimes_L M)_0(X) = (MU \otimes_L M)^0(X) \simeq \sum c_i \otimes m_i$, where $c_i \in MU^{d_i}(X)$ and $m_i \in M_{d_i}$. Thus $f$ factors as a composition

$$X \xrightarrow{(c_i)} \bigoplus \Sigma^{d_i} MU \xrightarrow{(m_i)} E$$

Therefore it suffices to prove $MU$ is evenly generated. Then, since $MU \simeq \varinjlim MU(n)$, it suffices to prove each $MU(n)$ is evenly generated (recall that the finite spectra are the compact objects). Recall $MU(n) = \mathrm{Th}(\eta^n - \varepsilon^n)$, where $\eta^n$ is the universal bundle, and $\varepsilon^n$ is the trivial bundle of rank $n$ on $BU(n)$. Finally, since $BU(n) \simeq \varinjlim Gr_n(\mathbb{C}^{n+k})$, by the Thom isomorphism, it suffices to prove the thom spectra of each $Gr_n(\mathbb{C}^{n+k})$ is an even spectrum, which is obvious by the Bruhat decomposition of $Gr_n$ into even dimensional cells. $\square$

**Proposition 11.1.5.** *Let $E$ be an evenly generated spectrum and $E'$ be any spectrum whose homotopy groups are concentrated in even degrees. Then every phantom map $E \to E'$ is null.*

*Proof.* We let $A$ denote the set of all finite even spectra equipped with a map to $E$, say $u_\alpha : X_\alpha \to E$. Then we consider the fiber

$$K \xrightarrow{\iota} \bigoplus X_\alpha \xrightarrow{u=\bigoplus u_\alpha} E$$

Then $u$ is classified by a map $u' : E \to \Sigma K$. Since $E$ is evenly generated, every map from a finite spectrum $X \to E$ factors through $u$, hence $X \to E \to \Sigma K$ is null. Thus $u'$ is a phantom. Conversly, if $f : E \to E'$ is any phantom map, then $f \circ u$ is null, hence $f$ factors as a composition $E \to \Sigma K \to E'$. Thus it now suffices to prove every map $\Sigma K \to E'$ is null, that is, $E'^{-1}(K) = 0$.
Since the homotopy groups of $E'$ are concentrated in even degrees, the Atiyah-Hirzebruch spectral sequence shows that $E'^{-1}(X) \simeq 0$ for $X$ a finite even spectrum. Thus it suffices to show that $K$ is the retract of a direct sum of finite even spectra. Let $B$ denote the collection of triples $(\alpha, \alpha', f)$, where $\alpha, \alpha' \in A$ and $u$ ranges over all homotopy classes of maps $X_\alpha \to X_{\alpha'}$ commuting with $u_\alpha$ and $u_{\alpha'}$. For $\beta = (\alpha, \alpha', f)$, we let $Y_\beta = X_\alpha$. Then clearly we have a map $\varphi : \bigoplus Y_\beta \to \bigoplus X_\alpha$, given on each factor $Y_\beta$ by the difference map of $Y_\beta \simeq X_\alpha \hookrightarrow \bigoplus X_\alpha$ and $Y\beta \to X_{\alpha'} \hookrightarrow \bigoplus X_\alpha$. Let $F$ denote the cofiber of $\varphi$. Since $\bigoplus Y_\beta$ is obviously evenly generated, we have a map of fiber sequences

$$\begin{tikzcd}
\bigoplus Y_\beta \arrow[r] \arrow[d] & \bigoplus X_\alpha \arrow[r] \arrow[d] & F \arrow[d] \\
K \arrow[r] & \bigoplus X_\alpha \arrow[r] & E
\end{tikzcd}$$

We construct a map $E \to F$ right inverse to the map $F \to E$, so as to exhibit $K$ as the retract of $\bigoplus Y_\beta$. It suffices to construct a map between the homology theories $E_* \to F_*$. By passing to filtered colimits and Spanier-Whitehead duality, it suffices to construct for every finite spectrum $X$ and a map $f : X \to E$ a map $q(f) : X \to F$.
Since $E$ is evenly generated, $f$ factors as $X \to X_{\alpha'} \to E$ for some $\alpha'$. Then we define $q(f)$ to be the map $X \to X_{\alpha'} \to \bigoplus X_\alpha \to F$. We prove $f$ is well-defined. For another factorization $X \to X_{\alpha''} \to E$, let $Y$ denote the pushout $X_{\alpha'} \amalg_X X_{\alpha''}$. Then $Y$ is a finite spectrum, hence the map $Y \to E$ induced by our assumptions factors as $Y \to X_\alpha \to E$. Let $h'$ denote the composition $X_{\alpha'} \to Y \to X_\alpha$ and $h''$ similarly. Then $(\alpha', \alpha, h')$ and $(\alpha'', \alpha, h'')$ are elements of $B$. Thus the composite maps $X \to X_{\alpha'} \to \bigoplus X_\alpha \to F$ and $X \to X_{\alpha''} \to \bigoplus X_\alpha \to F$ can be identified with $X \to Y \to X_\alpha \to \bigoplus X_\alpha \to F$, hence $q$ is well-defined. $\square$

**Corollary 11.1.6.** *Let $E$ be a Landweber-exact spectrum, and let $E'$ be a spectrum such that $\pi_{2k+1}(E') = 0$. Then every phantom map $f : E \to E'$ is null-homotopic.*

**Lemma 11.1.7.** *Recall the definition of $\mathcal{L}$-twisted formal group laws from definition 10.1.10. Then the following conditions are equivalent for a $\mathcal{L}$-twisted formal group law $f$.*

1. *The associated formal group $\mathcal{G}_f$ is classified by a flat map $q : \operatorname{Spec} R \to \mathcal{M}_{FG}$.*
2. *The graded $L$-algebra $\bigoplus \mathcal{L}^{\otimes n}$ is Landweber-exact.*

*Proof.* (2) $\implies$ (1): The condition is equivalent to $\bigoplus \mathcal{L}^{\otimes n}$ flat over $\mathcal{M}_{FG}$ by theorem 10.3.11. Thus we obviously have $\mathcal{L}^{\otimes 0} \simeq R$ flat over $\mathcal{M}_{FG}$.
(1) $\implies$ (2): $\bigoplus \mathcal{L}^{\otimes n}$ is flat over $R$. $\square$

**Corollary 11.1.8.** *In the situation of the above lemma, we may define a spectrum $E_R$ given by $(E_R)_*(X) = MU_*(X) \otimes_L (\bigoplus \mathcal{L}^{\otimes n})$.*

*Proof.* Directly by corollary 10.3.6. □

**Construction 11.1.9.** Let $R = L$ and $\mathcal{L} = R$ be trivial. Then $\bigoplus \mathcal{L}^{\otimes n} \simeq L[\beta^{\pm 1}]$. Then the above construction gives a spectrum $E_L$ whose homology theory is given by $(E_L)_*(X) = MU_*(X)[\beta^{\pm 1}]$. We denote this spectrum by $MP$, the periodic complex cobordism spectra. In particular, $MP_0(X) \simeq \bigoplus MU_{2n}(X)$.
Moreover, the above construction generalizes as follows. We let $\mathcal{L} \simeq R$, then we define $(E_R)_*(X) = MU_*(X) \otimes_L R[\beta^{\pm 1}] = MP_*(X) \otimes_L R$.

**Construction 11.1.10.** The construction in the corollary can be reformulated as follows. Let $q : \operatorname{Spec} R \to \mathcal{M}_{FG}$ be a flat map. Then $E_R$ is uniquely determined by the definition $(E_R)_0(X) = q^*\mathcal{F}_X$, where $\mathcal{F}_X$ is the associated quasi-coherent sheaf of $X$ over $\mathcal{M}_{FG}$.

**Construction 11.1.11.** For a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} R' \arrow[rr, "f"] \arrow[dr, "q'"'] & & \operatorname{Spec} R \arrow[dl, "q"] \\ & \mathcal{M}_{FG} & \end{tikzcd}$$

where $q$ and $q'$ are flat, we have an obvious induced map in homology $(E_R)_*(X) \to (E_{R'})_*(X)$. Then by corollary 11.1.6, we have a uniquely determined map up to homotopy $f : E_R \to E_{R'}$.

**Proposition 11.1.12.** *Suppose we have flat maps $q : \operatorname{Spec} R \to \mathcal{M}_{FG}$ and $q' : \operatorname{Spec} R' \to \mathcal{M}_{FG}$, the smash product $E_R \wedge E_{R'}$ is homotopy equivalent to $E_B$, where $B$ fits in the following Cartesian diagram*

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Spec} R' \arrow[d, "q'"] \\ \operatorname{Spec} R \arrow[r, "q"'] & \mathcal{M}_{FG} \end{tikzcd}$$

*Proof.* Clearly $\operatorname{Spec} B$ is flat over $\mathcal{M}_{FG}$. For simplicity, we assume the formal groups over $R$ and $R'$ classified by $q$ and $q'$ are coordinatizable, given by maps $L \to R$ and $L \to R'$. Then, recall from the construction in the proof of proposition 10.3.5, we have the pullback

$$\begin{tikzcd} \operatorname{Spec} MP_0(MP) = \operatorname{Spec} L[b_0^{\pm 1}, b_1, \dots] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Spec} L \arrow[d] \\ \operatorname{Spec} L \arrow[r] & \mathcal{M}_{FG} \end{tikzcd}$$

Thus for general $R$ and $R'$, we have $B \simeq R \otimes_L MP_0(MP) \otimes_L R'$ by pulling back. Finally, let $X$ be any spectrum, we have

$$\begin{aligned} (E_R \wedge E_{R'})_0(X) \simeq& (E_R)_0(E_{R'} \wedge X) \\ \simeq& R \otimes_L MP_0(E_{R'} \wedge X) \\ \simeq& R \otimes_L (E_{R'})_0(MP \wedge X) \\ \simeq& R \otimes_L MP_0(MP \wedge X) \otimes_L R' \\ \simeq& R \otimes_L (MP \wedge MP)_0(X) \otimes_L R' \end{aligned}$$

Thus we have $E_R \wedge E_{R'} \simeq E_B$. Finish the general case after calculating the general pullback. □

**Corollary 11.1.13.** *For any flat map* $\operatorname{Spec} R \to \mathcal{M}_{FG}$*, there is a canonical multiplication* $E_R \wedge E_R \to E_R$ *making* $E_R$ *into a commutative and associative algebra in the homotopy category of spectra.*

*Proof.* We consider the pullback diagram

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \operatorname{Spec} R \arrow[d] \\ \operatorname{Spec} R \arrow[r] & \mathcal{M}_{FG} \end{tikzcd}$$

Then we have obviously a map $\operatorname{Spec} R \to \operatorname{Spec} B$. Then this induces a map $E_R \wedge E_R \simeq E_B \to E_R$. The commutativity and associativity is obvious. □

**Definition 11.1.14.** Recall from corollary 11.1.8, we have $\pi_n(E_R) = \mathfrak{g}^k$ for $n = 2k$ and 0 otherwise, where $\mathfrak{g}$ is the Lie algebra of the formal group. Then it makes sense to have the following definition.
Let $E$ be a ring spectrum, we say $E$ is even periodic if the following conditions are satisfied.

1. $\pi_{2k+1}(E) = 0$.
2. The pairing $\pi_2(E) \otimes_{\pi_0(E)} \pi_{-2}(E) \to \pi_0(E)$ is an isomorphism (in particular, $\pi_2(E)$ is an invertible $\pi_0(E)$-module $\mathcal{L}$ and $\pi_{2n}(E) = \mathcal{L}^{\otimes n}$).

**Construction 11.1.15.** For a even periodic ring spectrum $E$, it is automatically complex-orientable by the degeneration of the Atiyah Hirzebruch spectral sequence of $E^*(\mathbb{CP}^\infty)$ on the second page. Then we have the calculation

$$\begin{aligned} E^d(\mathbb{CP}^\infty) &\simeq (\pi_*(E)[[t]])^d \\ &\simeq \prod_{n=0}^{\infty} \pi_{2n-d}(E) t^n \\ &\simeq \prod_{n=0}^{\infty} \pi_{-d}(E) \otimes_{\pi_0(E)} \pi_{2n}(E) t^n \\ &\simeq \pi_{-d}(E) \otimes_{\pi_0(E)} E^0(\mathbb{CP}^\infty) \end{aligned}$$

Then recall that from the construction construction 9.2.6 we have a coordinatizable formal group over $\pi_*(E)$ given by $\operatorname{Spf} E^*(\mathbb{CP}^\infty) \simeq \operatorname{Spf} \pi_*(E)[[t]]$. By the above calculation, we have $E^*(\mathbb{CP}^\infty) \simeq E^0(\mathbb{CP}^\infty) \otimes_{\pi_0(E)} \pi_*(E)$. Then notice the comultiplication on $E^*(\mathbb{CP}^\infty)$ preserves dimensions, hence it restricts to a comultiplication on $E^0(\mathbb{CP}^\infty)$. Then, this gives a formal group structure on $\operatorname{Spf} E^0(\mathbb{CP}^\infty)$ over $\pi_0(E)$, which base changes to $\operatorname{Spf} E^*(\mathbb{CP}^\infty)$ over $\pi_*(E)$.

**Corollary 11.1.16.** *Let* $\mathcal{C}$ *be the category of pairs* $(R, \eta)$*, where* $R$ *is a ring and* $\eta : \operatorname{Spec} R \to \mathcal{M}_{FG}$ *is a flat map. Then the construction* $R \mapsto E_R$ *determines a fully faithful embedding of* $\mathcal{C}$ *into the category of commutative algebras in the category of spectra. Moreover, a ring spectrum* $E$ *belongs to the image iff* $E$ *is even periodic, and the map* $\pi_0(E) \to \mathcal{M}_{FG}$ *is flat.*

*Proof.* The construction $E \mapsto (\pi_0(E), \operatorname{Spf} E^0(\mathbb{CP}^\infty))$ is a left inverse for this operation. Conversely, if $E$ is an even periodic ring spectrum, let $R = \pi_0(E)$, then the homology theory $E_R$ is given by $(E_R)_*(X) = MU_*(X) \otimes_L (\pi_*(E))$. Then we have a map $MU \to E$ given by the complex orientation on $E$, and since $E_*(X)$ is always a $\pi_*(E)$ module, we have a map $(E_R)_*(X) \to E_*(X)$. This map is an isomorphism when $X = *$, and also, recall from construction 11.1.11 that we have a map of spectra $E_R \to E$ well-defined up to homotopy equivalence. The two information induces an isomorphism $\pi_*(E_R) \to \pi_*(E)$, which gives an equivalence of spectra. Then finally it is obvious that this equivalence is compatible with the ring spectra structures. □

## 11.2 $E(n)$ and $K(n)$

**Construction 11.2.1.** Recall the formal group law over $W(k)[[v_1, \ldots, v_{n-1}]]$ in construction 10.5.11 exhibits it as a Landweber-exact, since $v_0 = p, v_1, \ldots, v_{n-1}$ is regular by construction and $v_n$ is invertible in $W(k)[[v_1, \ldots, v_{n-1}]]/(v_0, v_1, \ldots, v_{n-1}) \simeq k$ by the assumption that $f$ has height exactly $n$. Thus, by theorem 10.3.11 and corollary 11.1.8, we may construct a spectra $E(n)$ with $\pi_*(E(n)) = W(k)[[v_1, \ldots, v_{n-1}]][\beta^{\pm 1}]$. Notice the construction depends not only on $n$ but also a choice of $k$ and a formal group law of height $n$ on $k$.

Actually, we have a more intuitive way to view $E(n)$. We consider the property of $L_{E(n)}$.

**Theorem 11.2.2.** *A spectrum $X$ is $E(n)$-acyclic (recall the definition from definition 6.1.6) iff the associated sheaf $\mathcal{F}_X$ and $\mathcal{F}_{\Sigma X}$ over $\mathcal{M}_{FG} \times \operatorname{Spec} \mathbb{Z}_p$ are supported on the closed substack $\mathcal{M}_{FG}^{\geq n+1}$.*

*Proof.* No idea. □

This implies that the localization with respect to $E(n)$ has the intuition of restricting $\mathcal{F}_X$ to the substack $\mathcal{M}_{FG}^{\leq n}$. In the rest of this section, we attempt to construct the spectra whose associated localization that restricts $\mathcal{F}_X$ to $\mathcal{M}_{FG}^{n}$.

**Construction 11.2.3.** Consider the $p$-local complex cobordism spectrum $MU_{(p)}$. This spectrum is an $E^\infty$-ring, so $MU_{(p)}$-modules and relative smash products $M \wedge_{MU_{(p)}} N$ makes sense in a good way. By definition, we have $\pi_*(MU_{(p)}) \simeq L_{(p)} \simeq \mathbb{Z}_{(p)}[[t_1, \ldots]]$. Again, we let $t_{p^k - 1} = v_k$ and $t_0 = p$. We define $M(k)$ to be the cofiber of the map $\Sigma^{2k}(MU)_{(p)} \to MU_{(p)}$ given by multiplication by $t_k$.

**Lemma 11.2.4.** *Each $M(k)$ is an $A_\infty$ $MU_{(p)}$-algebra.*

*Proof.* We let the unit of $M(k)$ to be the evident map $u : MU_{(p)} \to M(k)$. Then the $n$-fold smash product $M(k) \wedge_{MU_{(p)}} \cdots \wedge_{MU_{(p)}} M(k)$ can be realized as the total homotopy cofiber (that is, the cofiber of the map from the colimit of the diagram without the conepoint to the conepoint) of the commutative diagram

$$\begin{tikzcd}
& \cdots \arrow[dl] & & \cdots \arrow[dl] \\
\Sigma^{4k} MU_{(p)} \arrow[rr] \arrow[dd] & & \Sigma^{2k} MU_{(p)} \arrow[dd] & \\
& \cdots \arrow[dl] & & \cdots \arrow[dl] \\
\Sigma^{2k} MU_{(p)} \arrow[rr] & & MU_{(p)} &
\end{tikzcd}$$

which is shaped like an $n$-dimensional cube with each vertex $\Sigma^{2nk} MU_{(p)}$, and the map between vertices given by multiplying with $t_k$. We prove that these information together indeed organizes into a simplicial object in the category of $MU_{(p)}$-modules.
We first give the multiplication map $M(k) \wedge_{MU_{(p)}} M(k) \to M(k)$. We let $K$ denote the cofiber of the map $(\Sigma^{2k} M(k))^2 \to M(k)$ in which the morphism is given by $(t_k, t_k)$. Or equivalently, we define $K$ to be the total cofiber of the following diagram.

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & \Sigma^{2k} MU_{(p)} \arrow[d] \\
\Sigma^{2k} MU_{(p)} \arrow[r] & MU_{(p)}
\end{tikzcd}$$

Then we have a map $\alpha : K \to M(k)$ given by the morphisms of fiber sequences

$$\begin{tikzcd} (\Sigma^{2k} MU_{(p)})^2 \arrow[r, "(t_k, t_k)"] \arrow[d, "(\mathrm{id},\mathrm{id})"'] & MU_{(p)} \arrow[r] \arrow[d, "\simeq"'] & K \arrow[d, dashed, "\alpha"] \\ \Sigma^{2k} MU_{(p)} \arrow[r, "t_k"'] & MU_{(p)} \arrow[r] & M(k) \end{tikzcd}$$

and $\beta : K \to M(k) \wedge_{MU_{(p)}} M(k)$ given by the morphisms of squares. Then we attempt to factor $\alpha$ through $\beta$, which would give a multiplication map. Notice that $\alpha$ factors through $\beta$ iff the composition $\mathrm{fib}(\beta) \to K \to M(k)$ is null-homotopic. To see this, we notice that $\mathrm{fib}(\beta)$ is equivalently the total cofiber of the diagram (the fiber of the morphism between the diagram associated to $K$ and to $M(k) \wedge_{MU_{(p)}} M(k)$)

$$\begin{tikzcd} \Sigma^{4k} MU_{(p)} \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & * \end{tikzcd}$$

which implies $\mathrm{fib}(\beta) \simeq \Sigma^{4k+1} MU_{(p)}$. Then, notice that since $M(k)$ is a $MU_{(p)}$-module, a map $\Sigma^{4k+1} MU_{(p)} = 0$ iff its corresponding element in $\pi_{4k+1}(M(k))$ is 0, which is obvious since $M(k)$ has homotopy groups concentrated in even degrees. Finish the proof. □

*Remark* 18. When we give the ring structure on $M(k)$, we have used the fact that $\pi_{4k+1}(M(k)) = 0$ to imply the corresponding element of the map $\mathrm{fib}(\beta) \simeq \Sigma^{4k+1} MU_{(p)}$ is null-homotopic. However, this null-homotopy corresponds to an element of $\pi_{4k+2} MU_{(p)}$, and thus the collection of multiplications of $M(k)$ form a $\pi_{4k+2}(M(k))$ torsor. Then, we consider the action of $\Sigma_2$ on the torsor (given by the action on $M(k) \wedge_{MU_{(p)}} M(k)$). Then we have $H^1(\Sigma_2, \pi_{4k+2}(M(k))) = 0$, hence when $p \neq 2$, the multiplication on $M(k)$ can be chosen to be homotopy commutative (i.e, $E_2$).

**Definition 11.2.5.** We define $K(n)$ to be the smash product

$$\bigwedge_{k \neq p^n - 1} {}_{MU_{(p)}} M(k) \wedge_{MU_{(p)}} MU_{(p)}[v_n^{-1}]$$

By lemma 11.2.4, we see that $K(n)$ has an $A_\infty$ algebra structure, and when $p \neq 2$, an $E_2$ structure.

**Corollary 11.2.6.** *By construction, we have*

$$\pi_*(K(n)) \simeq (\pi_*(MU_{(p)}))[v_n^{-1}]/(t_0, t_1, \ldots, t_{p^n-2}, t_{p^n}, \ldots) \simeq \mathbb{F}_p[v_n^{\pm 1}]$$

*Moreover, for a finite $p$-local spectra $X$, we can use this fact to compute the $K(n)$-homology of $X$ for $n \gg 0$ easily. Since $X$ is finite and $p$-local, $H_*(X; \mathbb{F}_p)$ vanishes in almost all degrees. In particular, we may suppose $H_k(X; \mathbb{F}_p) \neq 0$ implies $a \leq k \leq b$. Then for $n \geq b - a$, the Atiyah-Hirzebruch spectral sequence of $K(n)_*(X)$ degenerates, giving the fact that $K(n)_*(X) \simeq H_*(X; \mathbb{F}_p)[v_n^{\pm 1}]$ (here $v_n$ has dimension $2(p^n - 1)$, as usual).*

The localization with respect to $K(n)$ has the intuition of restricting to the strata $\mathcal{M}_{FG}^n$. We will show this by calculating the Bousfield class of $K(n)$.

**Theorem 11.2.7.** *$E(n)$ is Bousfield equivalent to $E(n-1) \wedge K(n)$. By convention, $E(0) = H\mathbb{Q}[\beta^{\pm}]$, which is Bousfield equivalent to $H\mathbb{Q}$. That is, a spectrum is $E(n)$-acyclic iff it is both $E(n-1)$-acyclic and $K(n)$-acyclic.*

*Proof.* We first notice that every $p$-local complex oriented cohomology theory $E$ having $\operatorname{Spec}\pi_*(E) \to \mathcal{M}_{FG} \times \operatorname{Spec}\mathbb{Z}_{(p)}$ a flat covering of $\mathcal{M}_{FG}^{\leq n}$ is Bousfield equivalent to $E(n)$ by theorem 11.2.2. This allows us to construct another representative of the Bousfield class of $E(n)$. For $m \leq n$, we define

$$Z(m) := \bigwedge_{k\geq 0, k\neq p^{m'}-1, m\leq m'\leq n} {}_{MU_{(p)}} M(k) \wedge_{MU_{(p)}} MU_{(p)}[v_n^{-1}]$$

By definition, $Z(n) = K(n)$, and $Z(0)$ is Bousfield equivalent to $E(n)$. Then, let $X$ be an $E(n)$-acyclic spectrum. Since $\mathcal{M}_{FG}^{\geq n+1} \subseteq \mathcal{M}_{FG}^{\geq n}$, it is $E(n-1)$-acyclic, by theorem 11.2.2. Then, since $Z(0)$ is Bousfield equivalent to $E(n)$, $X \wedge Z(0) \simeq *$. Then, since $X \wedge K(n) \simeq X \wedge Z(n) = (X \wedge Z(0)) \wedge_{MU_{(p)}} \bigwedge_{0\leq k<n} {}_{MU_{(p)}} M(p^k - 1)$, we have $X \wedge K(n) \simeq *$. Thus $X$ is $K(n)$ acyclic. Conversely, suppose $X$ is both $K(n)$-acyclic and $E(n-1)$-acyclic. We prove by induction that $X$ is $Z(i)$-acyclic assuming it is $Z(i+1)$-acyclic. Notice that we have a fiber sequence

$$\Sigma^{2(p^i-1)} Z(i) \xrightarrow{v_i} Z(i) \to Z(i+1)$$

Thus by smashing $X$ to this fiber sequence, we have $v_i$ acts invertibly on $Z(i) \wedge X$. Thus $Z(i) \wedge X \simeq Z(i)[v_i^{-1}] \wedge X$. Thus it suffices to prove $X$ is $Z(i)[v_i^{-1}]$-acyclic. But then, since

$$Z(i) = Z(0) \wedge_{MU_{(p)}} \bigwedge_{0\leq k<i} {}_{MU_{(p)}} M(p^k - 1)$$

it suffices to prove $X$ is $Z(0)[v_i^{-1}]$-acyclic, which is Bousfield equivalent to $E(i)$. But $i < n$, hence this follows from the assumption. $\square$

Then, we have the fact that $E(n-1)$ and $K(n)$ are somewhat disjoint. To prove this, we first need a highly non-trivial fact (which we will prove later), theorem 12.1.9.

**Theorem 11.2.8.** *Let $X$ be any spectrum. Then $L_{E(n-1)}(X)$ is $K(n)$-acyclic.*

*Proof.* By theorem 12.1.9, $L_{E(n-1)}$ is smashing, hence $L_{E(n-1)}(X) \wedge K(n) \simeq X \wedge L_{E(n-1)}(K(n))$. Thus it suffices to show $L_{E(n-1)}(K(n)) \simeq *$. In other words, it suffices to prove $E(n-1) \wedge K(n) \simeq *$. This is obvious since $E(n-1) \wedge K(n)$ is complex orientable, and its associated formal group law must be both of height $\leq n-1$ and of height $n$. $\square$

We now present a characterization of the $E(n)$-localization with the help of the two preceding theorems.

**Theorem 11.2.9.** *For any spectrum $X$, the pullback diagram*

$$\begin{tikzcd} X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & L_{K(n)}(X) \arrow[d] \\ L_{E(n-1)}(X) \arrow[r] & L_{E(n-1)}(L_{K(n)}(X)) \end{tikzcd}$$

*equipped with the evident map $X \to X'$ exhibits $X'$ as the $E(n)$-localization of $X$.*

*Proof.* Since $E(n)$-acyclic implies $E(n-1)$-acyclic and $K(n)$-acyclic, $E(n-1)$-local and $K(n)$-local implies $E(n)$-local. Then, since $E(n)$-local spectra are closed under pullbacks, $X'$ is $E(n)$-local. It now suffices to prove the map $X \to X'$ is an $E(n)$-equivalence. By theorem 11.2.7, it suffices to show that $X \to X'$ induces both a $K(n)$-equivalence and an $E(n-1)$-equivalence. In other words, we must show that the squares

$$\begin{tikzcd} X \wedge K(n) \arrow[r] \arrow[d] & L_{K(n)}(X) \wedge K(n) \arrow[d] \\ L_{E(n-1)}(X) \wedge K(n) \arrow[r] & L_{E(n-1)}(L_{K(n)}(X)) \wedge K(n) \end{tikzcd}$$

$$\begin{tikzcd} X \wedge E(n-1) \arrow[r] \arrow[d] & L_{K(n)}(X) \wedge E(n-1) \arrow[d] \\ L_{E(n-1)}(X) \wedge E(n-1) \arrow[r] & L_{E(n-1)}(L_{K(n)}(X)) \wedge E(n-1) \end{tikzcd}$$

are pullback squares. Here, the second diagram obviously has two vertical morphisms by definition equivalences, hence is a pullback square. Now, the first square has the top horizontal morphism an equivalence, hence it suffices to prove the bottom morphism is an equivalence. But by theorem 11.2.8, the bottom line is the zero morphism between $*$, hence an equivalence. □

**Construction 11.2.10.** If $X$ is an $E(n)$-local spectrum, the above argument shows that $X \simeq L_{E(n-1)}(X) \vee_{L_{E(n-1)}(L_{K(n)})(X)} L_{K(n)}(X)$. Conversely, if we have a $K(n)$-local spectrum $Y$ and an $E(n-1)$-local spectrum $Z$, we can form a pullback diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d] \\ Z \arrow[r] & L_{E(n-1)}(Y) \end{tikzcd}$$

Then $X$ is $E(n)$-local. Moreover, $Y$ can be identified with $L_{K(n)}(X)$, and $Z$ can be identified with $L_{E(n-1)}(X)$. This implies that the category of $E(n)$-local spectra is equivalent to the category of triples $(Y, Z, \alpha : Z \to L_{E(n-1)}(Y))$.

## 11.3 Fields and uniqueness of $K(n)$

Notice that in the construction of $M(k)$, the choice of the multiplication of $M(k)$ is not canonically determined. Moreover, since the choice of $t_k$ for $k \neq p^i - 1$ is not canonical, the underlying spectrum of $M(k)$ is also not canonically determined. However, when they are put together, surprisingly we have the $K(n)$'s are independent of these choices, though the ring structures on them do depend on the ring structures of $M(k)$.

**Definition 11.3.1.** Let $R$ be a commutative evenly graded ring. We say that $R$ is a graded field if every non-zero homogeneous element of $R$ is invertible. Equivalently, $R$ is a field iff $R$ is a field in the ordinary sense concentrated at degree 0 or of the form $k[\beta^{\pm 1}]$ for some $\beta$ of positive even degree. We say a ring spectrum $E$ is a field if $\pi_*(E)$ is a field in the above sense.

**Corollary 11.3.2.** *For a field $R$, any graded module $M$ over $R$ admits a free basis of homogeneous elements.*

*Proof.* When $R$ is a field in the ordinary sense, this is obvious. When $R = k[\beta^{\pm 1}]$ and $\beta$ has degree $d$, then a $k$-basis of $\bigoplus_{0 \le i < d} M_i$ is a $R$-basis of $M$. □

**Corollary 11.3.3.** *For a ring spectrum $E$ which is a field, every $E$-module $M$ is of the form $\bigvee_\alpha \Sigma^{k_\alpha} E$.*

*Proof.* Simply pick a homogeneous $\pi_*(E)$ basis of $\pi_*(M)$, and such a basis determines a map $\bigvee_\alpha \Sigma^{k_\alpha} E \to M$, which is obviously a homotopy equivalence. □

**Corollary 11.3.4.** *For spectra $X$ and $Y$ and a field $E$, we have $E_*(X \wedge Y) \simeq E_*(X) \otimes_{\pi_*(E)} E_*(Y)$.*

*Proof.* Notice we have

$$\begin{aligned} & E \wedge X \wedge Y \\ =& (\bigvee_\alpha \Sigma^{k_\alpha} E) \wedge Y \\ =& \bigvee_\alpha \Sigma^{k_\alpha}(E \wedge Y) \\ =& \bigvee_\alpha \Sigma^{k_\alpha}(\bigvee_\beta \Sigma^{\ell_\beta} E) \\ =& \bigvee_{\alpha,\beta} \Sigma^{k_\alpha+\ell_\beta} E \end{aligned}$$

Then taking homotopy groups gives the results. □

**Corollary 11.3.5.** *$K(n)$ is a field. In particular, $K(0) \simeq H\mathbb{Q}$ and $K(\infty) \simeq H\mathbb{F}_p$ are fields.*

**Lemma 11.3.6.** *Let $f : X \to Y$ be a map of spectra. Then suppose that $X$ and $Y$ are $K(n)$-modules, we have $f_* : \pi_*(X) \to \pi_*(Y)$ is $\pi_*(K(n)) = \mathbb{F}_p[v_n^{\pm}]$-linear.*

*Proof.* We factor the map as a composition $X \to K(n) \wedge X \to K(n) \wedge Y \to Y$, where the middle spectra are regarded as a $K(n)$-module by the multiplication on $K(n)$. The second and third maps are obviously $K(n)$-module maps, hence it suffices to treat the case where $f$ is of the form $f : X \to K(n) \wedge X$. Since $K(n)$ is a field, $X$ is free, hence it suffices to prove the case for $X = K(n)$. Thus we must prove that the inclusion on the two factors $f, g : K(n) \to K(n) \wedge K(n)$ induces the same map on homotopy groups. In other words, we must show $f_*(v_n) = g_*(v_n)$. Now that $K(n) \wedge K(n)$ has two complex orientations obtained from each component, we have two formal group laws over $R = \pi_{even}(K(n) \wedge K(n))$. Obviously these formal group laws has $[p]$-series beginning with $f_*(v_n)t^{p^n}$ and $g_*(v_n)t^{p^n}$ by definition. Then, since the formal group laws differ by a coordinate change of the form $t \mapsto t + b_1 t^2 + \cdots$, we have $f_*(v_n) = g_*(v_n)$. □

**Corollary 11.3.7.** *For $X$ a spectrum admitting the structure of a $K(n)$-module. Then any retract of $X$, say $Y$, has the structrue of a $K(n)$-module.*

*Proof.* Consider the composition

$$Y \xrightarrow{i} X \xrightarrow{r} Y \xrightarrow{i} X$$

By definition, the composition of the first two maps is the identity $\mathrm{id}_Y$, and the composition of the last two maps is a map $f : X \to X$. By lemma 11.3.6, $f_*$ is a $\mathbb{F}_p[v_n^{\pm 1}]$-module map from $\pi_*(X)$ to itself. Thus the image of $f_*$ is a graded submodule of $\pi_*(X)$, and has a basis of classes $\eta_\alpha \in \pi_{k_\alpha} X$. Then clearly we have $Y \simeq \bigvee \Sigma^{k_\alpha} K(n)$. □

**Corollary 11.3.8.** *Let $E$ be any field with $E \wedge K(n) \neq 0$. Then $E$ admits the structure of a $K(n)$-modules.*

*Proof.* Notice the unit map of $E$-module structure $E \to E \wedge K(n)$ is non-zero, hence $E$ is a direct summand of $E \wedge K(n)$ since $E$ is a field. Thus by the preceding corollary, we have $E$ a retract of $E \wedge K(n)$, hence admits a $K(n)$-module structure. □

**Corollary 11.3.9.** *Let $E$ be a complex oriented cohomology theory having a formal group law of height exactly $n$. If $E \neq 0$, then $E \wedge K(n) \neq 0$.*

*Proof.* Notice $E \wedge E(n-1)$ is a complex oriented cohomology theory having formal group law both of height $\leq n-1$ and exactly $n$, hence is 0. If $E \wedge K(n) = 0$, then $E \wedge E(n) = 0$ by construction 11.2.10. Thus the quasi-coherent sheaf on $\mathcal{M}_{FG}$ associated to $E$, $\mathcal{F}_E$, vanishes when we restrict to the open substack $\mathcal{M}_{FG}^{\leq n}$. Since the formal group law on $E \wedge E(n)$ has height $\leq n$, we have $E \simeq 0$. □

**Theorem 11.3.10.** *Let $E$ be any complex oriented ring spectrum whose formal group law is of height exactly $n$, and whose homotopy groups are given by $\pi_*(E) \simeq \mathbb{F}_p[v_n^{\pm 1}]$. Then we have an equivalence of spectra $E \simeq K(n)$.*

*Proof.* It follows from the preceding two corollaries directly that $E$ is a free $K(n)$-module. By considering $\pi_*(E)$ we see this module is of rank 1, hence $E \simeq K(n)$. □

Moreover, we actually have any field $E$ is a $K(n)$-module for some $n$. To prove this, we need some preliminary results.

**Lemma 11.3.11.** *Let $\{E^\alpha\}$ denote a collection of ring spectra. The following conditions are equivalent.*

1. *Let $R$ be a $p$-local ring spectrum. If $x \in \pi_m(R)$ is a homotopy class whose image in $E_0^\alpha(R)$ is zero for all $\alpha$, then $x$ is nilpotent in $\pi_*(R)$.*

2. *Let $R$ be a $p$-local ring spectrum. If $x \in \pi_0(R)$ is a homotopy class whose image in $E_0^\alpha(R)$ is zero for all $\alpha$, then $x$ is nilpotent in $\pi_0(R)$.*

3. *Let $X$ be any $p$-local spectrum. If $x \in \pi_0(X)$ has trivial image under the Hurewicz map $\pi_0(X) \to E_0^\alpha(X)$ for all $\alpha$, then the induced class $x^{\otimes n}$ in $\pi_0(X^{\wedge n})$ is zero for $n \gg 0$.*

4. *Let $X$ be any $p$-local spectrum, and let $F$ be a finite spectrum. If $f : F \to X$ is such that each composite map $F \to X \to X \wedge E^\alpha$ is null, then $f^{\otimes n} : F^{\wedge n} \to X^{\wedge n}$ is null for $n \gg 0$.*

*Proof.* (1) $\Longrightarrow$ (2) is obvious, and (2) $\Longrightarrow$ (3) by taking $R$ to be $\bigvee_n X^{\wedge n}$. (3) $\Longrightarrow$ (4) by considering the function spectrum $\mathrm{Map}(F, X)$. Finally, suppose (4) holds. Then let $x \in \pi_m(R)$ be a class whose image vanishes in $E_m^\alpha(R)$ for all $\alpha$. We identify $x$ with a map $\Sigma^m S \to R$. Then $x^n$ can be identified with the composition $\Sigma^{mn} S \to R^{\wedge n} \to R$, where the second map is given by the multiplication. Since $x^{\otimes n}$ is null for $n \gg 0$, $x$ is nilpotent. □

**Definition 11.3.12.** We say a collection of ring spectra $\{E^\alpha\}$ detects nilpotence if the above equivalent conditions holds.

Recall the theorem lemma 8.2.3. Then we have the following corollary.

**Corollary 11.3.13.** *For an element $x \in \pi_0(X)$, we denote $x^{-1}X$ the colimit $\varinjlim(S \xrightarrow{x} X \xrightarrow{x} X^{\wedge 2} \to \cdots)$. Then let $E$ be any ring spectrum. Then the $E$-acyclicity of $T$ is equivalent to the condition that $x^{\otimes n}$ vanishes in $E_0(X^{\wedge n})$ for $n \gg 0$.*

*Proof.* Consider the ring spectrum $\bigvee X^{\wedge n}$. □

Recall that we have the following theorem.

**Theorem 11.3.14.** *The spectrum $MU$ detects nilpotence.*

This implies the following.

**Theorem 11.3.15.** *The collection $\{K(n)\}$ detects nilpotence.*

*Proof.* We verify the third equivalent condition. Now, we fix $x \in \pi_0(X)$ such that its image in $K(n)_0(X)$ is trivial for all $n$. We wish to prove that for some $n$ we have $x^{\otimes n}$ is trivial. By theorem 8.2.1, it suffices to check that the the image of $x$ in $MU_0(X)$ is nilpotent. Thus, it suffices to prove that $MU_*(T) = 0$. That is, we must show the quasi-coherent sheaf $\mathcal{F}_{\Sigma^k T}$ on $\mathcal{M}_{FG}$ vanishes for $k \in \mathbb{Z}$.
We consider the fiber sequences $\Sigma^{2k} MU_{(p)} \xrightarrow{t_k} MU_{(p)} \to M(k)$. For $n \geq 0$, we let $P(n)$ denote the smash product

$$\bigwedge_{k \neq p^m - 1 (m \geq n)} {}_{MU_{(p)}} M(k)$$

Thus we have $\pi_*(P(n)) = \mathbb{Z}_{(p)}[v_1, \ldots]/(v_0, v_1, \ldots, v_{n-1})$. Therefore $P(0) = BP$ (recall the definition of $BP$ in proposition 10.3.7), which is Landweber-exact. Thus the map $\operatorname{Spec} \pi_*(P(0)) \to \mathcal{M}_{FG} \times \operatorname{Spec} \mathbb{Z}_{(p)}$ is faithfully flat. Thus $P(0)_*(X)$ is the pullback of the quasi-coherent sheaf $\mathcal{F}_{\Sigma^{-*} X}$ on $\mathcal{M}_{FG}$. Thus it suffices to show $P(0)_*(T) = 0$. Let $P(\infty) = \varinjlim P(n)$, hence $P(\infty) = H\mathbb{F}_p$. Thus the image of $x$ in $P(\infty)_0(X)$ vanishes, hence the image of $x$ in $P(n)_0(X)$ vanishes for some $n$. Thus, $P(n)_0(T) \simeq *$.
We prove $P(m)_0(T) \simeq *$ for all $m$ using induction. Assume $P(m+1)_0(T) \simeq *$. Then we have the fiber sequence $\Sigma^{2(p^m - 1)} P(m) \xrightarrow{v_m} P(m) \to P(m+1)$. Thus the multiplication of $v_m$ is invertible on $P(m)_*(T)$, thus $P(m)_*(T) \simeq P(m)[v_m^{-1}]_*(T)$. Thus it suffices to prove $T$ is $MU_{(p)}[v_m^{-1}]$-acyclic. Notice that this is a Landweber-exact theory whose associated formal group law has height $\leq m$, hence it suffices to prove $T$ is $E(m)$-acyclic. However, since $T$ is $K(k)$-acyclic, by assumption, this follows immediately. □

**Corollary 11.3.16.** *For a nonzero $p$-local ring spectrum $E$, $E \wedge K(n)$ is nonzero for some $n$.*

*Proof.* If not, then every element of $\pi_0(E)$ is nilpotent, in particular, the unit, which implies $E \simeq *$. □

**Corollary 11.3.17.** *Let $E$ be a ring spectrum such that $\pi_*(E)$ is a graded field. Then $E$ has the structure of a $K(n)$-module for some $p$ and $n$.*

*Proof.* Take $p$ to be the characteristic of $\pi_*(E)$. □

## 11.4 The periodicity theorem

Instead of arbitrary spectra, we consider the category of $p$-local spectra in this section.

**Definition 11.4.1.** Let $\mathcal{T}$ be a full subcategory of some stable category $\mathcal{C}$. We say $\mathcal{T}$ is thick if it contains 0, is closed under the formation of fibers and cofibers, and if every retract of a spectrum in $\mathcal{T}$ is in $\mathcal{T}$.

**Construction 11.4.2.** By definition the intersection of two thick subcategories is still thick. So for a collection $\mathcal{F}$ of objects in $\mathcal{C}$, it makes sense to talk of the minimal thick subcategory containing $\mathcal{F}$, denoted by $\mathcal{T}(\mathcal{F})$. This subcategory will be called the thick subcategory generated by $\mathcal{F}$, and is indeed generated by $\mathcal{F}$ in the following sense.

Let $\mathcal{F}'$ be the collection of objects in $\mathcal{C}$ which is a suspension of objects in $\mathcal{F}$. Then let $\mathcal{T}_0(\mathcal{F}') = 0$, and $\mathcal{T}_1(\mathcal{F}')$ denote the full subcategory spanned by objects $\tilde{X}$ which are retracts of objects $X \in \mathcal{F}$. Then inductively, we define $\mathcal{T}_n(\mathcal{F}')$ to be the full subcategory spanned by objects $\tilde{X}_n$ which are retracts of objects $X_n$, and $X_n$ sits in a fiber sequence $Y \to X_n \to Z$, where $Y \in \mathcal{T}_1 \mathcal{F}'$ and $Z \in \mathcal{T}_{n-1}(\mathcal{F}')$. Finally we let $\mathcal{T}(\mathcal{F}) = \bigcup \mathcal{T}_n(\mathcal{F}')$. This is clearly the minimal thick subcategory containing $\mathcal{F}$.

Moreover, the above construction has the obvious property that for $Y \in \mathcal{T}_{n_1}(\mathcal{F})$ and $Z \in \mathcal{T}_{n_2}(\mathcal{F})$, if we have a fiber sequence $Y \to X \to Z$, then $X \in \mathcal{T}_{n_1+n_2}(\mathcal{F})$.

**Lemma 11.4.3.** *For any thick subcategory $\mathcal{T}$, and let $X \in \mathcal{T}$. Then for any $p$-local spectrum $Y$, we have $X \wedge_{S_{(p)}} Y \in \mathcal{T}$.*

*Proof.* The subcategory consisting of $Y$ such that $X \wedge_{S_{(p)}} Y \in \mathcal{T}$ is also thick, and contains $S_{(p)}$ by definition. Thus this subcategory is the whole category of finite $p$-local spectra, since finite $p$-local spectra admit finite cell decompositions. $\square$

**Construction 11.4.4.** For a thick subcategory $\mathcal{T}$, let $\mathcal{C}$ denote the collection of $p$-local spectra generated by $\mathcal{T}$ under colimits. By the argument of presentable categories and compact objects, every object $X \in \mathcal{C}$ is a filtered colimit of objects in $\mathcal{T}$, and in particular, if $X$ is a finite $p$-local spectrum, then the identity map $X \to \varinjlim X_\alpha$ factors through some $X_\alpha$ by the compactness. Then $X$ is the retract of $X_\alpha$ hence $X \in \mathcal{T}$. Thus the construction $\mathcal{T} \mapsto \mathcal{C}$ determines a bijection between thick subcategories of finite $p$-local spectra and subcategories of $\mathcal{C}$ of the category of $p$-local spectra stable under desuspension and generated by finite $p$-local spectra under colimits.

**Proposition 11.4.5.** *Let $\mathcal{C}$ denote a subcategory of $p$-local spectra stable under desuspension and generated by a thick subcategory $\mathcal{T}$ under colimits. Then the localization w.r.t $\mathcal{C}$ is smashing.*

*Proof.* It suffices to prove $\mathcal{C}^\perp$ is closed under colimits, by lemma 6.1.13. Let $X_\alpha \in \mathcal{C}^\perp$. Then for any $Y \in \mathcal{C}$, we prove $[Y, \varinjlim X_\alpha]_*$ is trivial. However, since $Y \in \mathcal{C}$, $Y = \varinjlim Y_\beta$ for $Y_\beta \in \mathcal{T}$. Then since $Y_\beta$ is finite, we have $[Y_\beta, \varinjlim X_\alpha] \simeq \varinjlim [Y_\beta, X_\alpha]$. Thus

$$\begin{aligned}
&[Y, \varinjlim X_\alpha]\\
\simeq&[\varinjlim Y_\beta, \varinjlim X_\alpha]\\
\simeq&\varprojlim [Y_\beta, \varinjlim X_\alpha]\\
\simeq&\varprojlim \varinjlim [Y_\beta, X_\alpha]\\
\simeq&0
\end{aligned}$$

$\square$

**Proposition 11.4.6.** *Let $X$ be a finite $p$-local spectrum. Suppose $K(n)_*(X) = 0$ for some $n \geq 0$. Then $K(n-1)_*(X) = 0$.*

*Proof.* We let $R$ denote the ring spectrum

$$\bigwedge_{k \neq p^n-1, p^{n-1}-1} {}_{MU_{(p)}} M(k) \wedge_{MU_{(p)}} MU_{(p)}[v_n^{-1}]$$

We assume $n > 1$ (when $n = 1$, the proof is essentially the same, with some minor changes to the notations). Then, $\pi_*(R) \simeq \mathbb{F}_p[v_{n-1}, v_n^{\pm 1}]$. In particular, $\pi_0(R)$ is equivalent to the polynomial ring $\mathbb{F}_p[v_{n-1}^a v_n^{-b}]$ where $(a, b)$ is the minimal solution to $a(p^{n-1}-1) = b(p^n - 1)$. Then, for every integer $k$, $R_k(X)$ is a finitely generated module over $\pi_0(R)$.
Consider the fiber sequence $\Sigma^{2(p^{n-1}-1)} R \xrightarrow{v_{n-1}} R \to K(n)$. Since $K(n)_*(X) = 0$, the multiplication by $v_{n-1}$ is invertible on $R_*(X)$. Thus, the multiplication of $v_{n-1}^a v_n^{-b} := t$ is invertible on each $R_k(X)$. Notice that if $R_k(X)$ has a $\mathbb{F}_p[t]$-free part, then $t$ is obviously not invertible, hence $R_k(X)$ is a torsion $\mathbb{F}_p[t]$-module. In particular, there is a polynomial $f$ such that $f$ annihilates each $R_k(X)$ for $0 \leq k < 2(p^n - 1)$, and since $\pi_*(R)$ is periodic of period $2(p^n-1)$, $f$ annihilates $\pi_*(R)$. Since $t$ has invertible action, we may assume $f$ is divisible by $t$. Then for $k \gg 0$, the product $f(t)v_n^k$ can be written as a polynomial in $v_{n-1}$ and $v_n$, and hence comes from $\pi_*(MU_{(p)})$. Thus multiplication by $f(t)$ on $R$ is well-defined, and we may localize $R$ to $R[f(t)^{-1}]$ with $R[f(t)^{-1}]_*(X) \simeq R_*(X)[f(t)^{-1}] = 0$.
Then, $R[f(t)^{-1}]$ has a complex orientation, which induces a formal group law of height exactly $n-1$ (since $f(t)$ is divisible by $t$, $v_{n-1}$ is invertible in $\pi_*(R[f(t)^{-1}])$). Thus $R[f(t)^{-1}] \wedge K(m)$

vanishes for $m \neq n-1$, and hence by corollary 11.3.16, $R[f(t)^{-1}] \wedge K(n-1) \neq 0$, and contains $K(n-1)$ as a retract. But $X \wedge R[f(t)^{-1}] \simeq 0$, thus $X \wedge R[f(t)^{-1}] \wedge K(n-1) \simeq 0$ and thus $X \wedge K(n-1) \simeq 0$. $\square$

**Definition 11.4.7.** We say a $p$-local spectrum $X$ has type $n$ if $K(n)_*(X) \neq 0$ but $K(m)_*(X) = 0$ for all $m < n$. By the preceding theorem, every spectrum is of type $n$ for some $n$. For exaxmple, $X$ has type 0 iff $H_*(X;\mathbb{Q}) \neq 0$, or equivalently iff $H_*(X;\mathbb{Z})$ is not torsion. From now on, we let $\mathcal{C}_{\geq n}$ denote the collection of finite $p$-local spectra of type $\geq n$.

**Lemma 11.4.8.** *$\mathcal{C}_{\geq n}$ are thick subcategories.*

*Proof.* It is obviously closed under the formation of fibers and cofibers by the long exact sequence of $K(m)$-homology. Then, the retracts of spectra induce retracts of homology, thus this category is also closed under retracts. $\square$

**Theorem 11.4.9.** *Every thick subcategory is of the form $\mathcal{C}_{\leq n}$ for some $n$.*

*Proof.* If $\mathcal{T} = 0$, then we have $\mathcal{T} = \mathcal{C}_{\leq\infty}$. Otherwise, we claim that the thick subcategory $\mathcal{T} = \mathcal{C}_{\leq n}$, where $n$ is the minimal $n$ such that there exists a spectrum of type $n$ in $\mathcal{T}$. We prove that for $X \in \mathcal{T}$ of type $n$, and $Y$ in $\mathcal{T}$ of type $\geq n$, then $Y \in \mathcal{T}$.
Let $DX$ denote the $p$-local Spanier-Whitehead dual of $X$. Then the identity map $X \to X$ is classified by a map $e : S_{(p)} \to X \wedge_{S_{(p)}} DX$. Since $X$ has type $n$, $e$ induces an injection for $m \geq n$ (by corollary 11.3.4)

$$K(m)_*(S_{(p)}) \to K(m)_*(X \wedge_{S_{(p)}} DX) \simeq K(m)_*(X) \otimes_{\mathbb{F}_p[v_m^{\pm 1}]} K(m)_*(X)^\wedge$$

here $K(m)_*(X)^\wedge$ denote the algebraic dual of $K(m)_*(X)$. We consider the fiber $F = \mathrm{fib}(S_{(p)} \to X \wedge DX)$. Thus the map $f : F \to S_{(p)}$ induces a zero map $K(m)_*(F) \to K(m)_*(S_{(p)})$ for $m \geq n$, and we may consider the composite $g : F \to S_{(p)} \to Y \wedge DY$. Then $g$ induces the zero map $K(m)_*(F) \to K(m)_*(Y \wedge DY)$ for $m \geq n$ since $f$ is the zero map, and also when $m < n$ since $Y$ is of type $n$. Thus by theorem 11.3.15, some smash product of $g^{\wedge k} : F^{\wedge k} \to (Y \wedge DY)^{\wedge k}$ is null. Then, since $Y \wedge DY$ is a ring spectra (with multiplication given by the counit map), we obtain a null-homotopic map $F^{\wedge k} \to Y \wedge DY$. By the definition of the counit maps, this map corresponds to the composition

$$F^{\wedge k} \wedge Y \xrightarrow{f} F^{\wedge k-1} \wedge Y \to \cdots \to Y$$

Thus $Y$ is a retract of the cofiber $Y/(F^{\wedge k} \wedge Y)$. Also, notice that $Y/(F^{\wedge k} \wedge Y)$ admit a finite filtration $(F^{\wedge a} \wedge Y)/(F^{\wedge a+1} \wedge Y)$. Thus it suffices to prove each $F^{\wedge a} \wedge Y/(F^{\wedge a+1} \wedge Y)$ is in $\mathcal{T}$, since it is thick. However, these spectra are of the form $F^{\wedge a} \wedge Y \wedge (S_{(p)}/F) \simeq F^{\wedge a} \wedge Y \wedge DX \wedge X$, and thus belongs to $\mathcal{T}$ since $X$ belongs to $\mathcal{T}$ and everything else are finite $p$-local, by lemma 11.4.3. $\square$

We have seen that all thick subcategories are of the form $\mathcal{C}_{\geq n}$. However, we still need to prove that the $\mathcal{C}_{\geq n}$ are distinct. That is, we need to construct for each $n$ a finite $p$-local spectrum $X$ of type $n$. However, this, along with a stronger assumption, is a part of a famous construction. We will use this fact without proof, later.

**Definition 11.4.10.** Let $X$ be a finite $p$-local spectrum, and let $n \geq 1$. Then we define a $v_n$-self map is a map $f : \Sigma^k X \to X$ such that $f$ induces an isomorphism $K(n)_*(X) \to K(n)_*(X)$, and for $m \neq n$, the induced map $K(m)_*(X) \to K(m)_*(X)$ is nilpotent.
Notice that $X \wedge DX$ carries a natural ring spectrum structure, and giving a $v_n$-self map of degree $k$ is simply giving an element in $\pi_k(X \wedge DX)$. Thus we define for a $p$-local ring spectrum, an element $x \in \pi_k(R)$ is an $v_n$-element iff the image of $x$ in $K(m)_*(R)$ is nilpotent for $m \neq n$, and invertible for $m = n$.
Also, notice that a $v_n$-element induces a $v_n$-self map of $R$ given by multiplication. In particular, this implies that $R$ has type $> 0$, hence has homotopy groups are torsion.

**Lemma 11.4.11.** *If $X$ is a finite $p$-local spectrum that admits a $v_n$-self map $f$, $X$ has type $\geq n$.*

*Proof.* If $X$ has type $m < n$, then since $f$ is not an isomorphism on $K(m)$-homology, we have $\mathrm{cof}(f) = X/f$ has non-vanishing $K(m)$-homology, but has vanishing $K(n)$-homology, which contradicts proposition 11.4.6. □

*Remark* 19. The zero map $0 : X \to X$ is a $v_n$-self map if $X$ is of type $> n$.

**Construction 11.4.12.** Let $X$ be a spectrum of type $n$. Then, let $f : \Sigma^k X \to X$ be a $v_n$-self map. We consider $X/f := \mathrm{cof}(f)$. By definition, $X/f$ is of type $\geq n$. However, since $f$ is a $v_n$-self map, it induces an isomorphism on $K(n)$-homology. Thus $X/f$ has trivial $K(n)$-homology, hence is of type $> n$. But $f$ does not induce an isomorphism on $K(n+1)$-homology by the definition of $v_n$-self maps, hence $X/f$ must be of type $n+1$.

**Theorem 11.4.13.** *Let $X$ be a finite $p$-local spectrum of type $\geq n$. Then $X$ admits a $v_n$-self map.*

Again, we need a few lemmas before the proof.

**Lemma 11.4.14.** *Let $R$ be a $\mathbb{Z}_{(p)}$-algebra, and let $x, y \in R$ be commuting elements such that $x - y$ is torsion and nilpotent. Then $x^{p^k} = y^{p^k}$ for $k \gg 0$.*

*Proof.* We have

$$x^{p^k} = (y + (x-y))^{p^k} = y^{p^k} + \sum_{i=1}^{p^k} \binom{p^k}{i} y^{p^k - i}(x-y)^i$$

Now, suppose $(x - y)$ is nilpotent of degree less than $p^a$ (that is, $(x-y)^{p^a} = 0$), the right hand side could be rewritten as

$$x^{p^k} = y^{p^k} + \sum_{i=1}^{p^a - 1} \frac{p^k}{i} \binom{p^k - 1}{i - 1} y^{p^k - i}(x-y)^i$$

Now, $p^k/i$ ($R$ is a $\mathbb{Z}_{(p)}$ algebra, so this makes sense) is divisible by $p^{k-a}$ for all $i$, hence for $k \gg 0$, $p^k(x-y)/i = 0$. □

**Lemma 11.4.15.** *Let $R$ be a finite $p$-local spectrum, and let $x \in \pi_k(R)$ be a $v_n$-element. Then a power of $x$ satisfies that $x \mapsto v_n^a \in K(n)_*(R)$, and $x \mapsto 0 \in K(m)_*(R)$ for $m \neq n$. This lemma strengthens the assumption of $v_n$-elements.*

*Proof.* Recall that by corollary 11.2.6 we have $K(m)_*(R) = H_*(R; \mathbb{F}_p)[v_m^{\pm 1}]$ for $m \gg 0$. Then, we have $x$ nilpotent in $H_*(R; \mathbb{F}_p)$. Hence for $m \gg 0$, we have a power of $x$ mapping to $0 \in K(m)_*(R)$. Then, there are finitely many remaining terms $k$ between $n$ and a sufficiently large number such that the image of $x$ in $K(k)_*(R)$ is nilpotent but not necessarily 0. Since there are only finitely many terms, we may replace $x$ by the degree of nilpotency one by one. Thus we have some $b$ such that $x^b \mapsto 0$ for all $m \neq n$. From now on, we let $x$ be $x^b$. Then, since $K(n)_*(R)$ is a finite module over $\pi_*(K(n))$, we have $K(n)_*(R)/(v_n - 1)$ is finite. But note that the image of $x$ in $K(n)_*(R)/(v_n - 1)$ is a unit, hence by finiteness a power of $x$ is 1 in $K(n)_*(R)/(v_n - 1) \simeq \bigoplus_{0 \leq i < 2(p^n - 1)} K(n)_i R$. Then by lifting the image back, we have a power of $x$ is the power of $v_n$. □

**Lemma 11.4.16.** *Let $R$ be a finite $p$-local ring spectrum, and let $x \in \pi_k(R)$ be an $v_n$-element. After raising $x$ to a suitable power, we may assume $x$ is central in $\pi_*(R)$.*

*Proof.* We first assume $x$ satisfies the additional properties of lemma 11.4.15. Let $A = R \wedge DR$, and let $a, b \in A$ be determined by the self maps of $R$ given by left and right multiplication by $x$. Then obviously $a$ and $b$ commute. Since $A$ is of type $> 0$, $a - b \in \pi_k A$ is torsion. Then, it suffices to prove that $a - b$ is nilpotent, thus by lemma 11.4.14, we have $a^{p^k} = b^{p^k}$, and by taking $x$ to be $x^{p^k}$, we see that the left multiplication and right multiplication coincide.
By theorem 11.3.15, it suffices to prove the image of $a - b$ vanishes in all $K(m)_*(A)$. In other words, we must prove the compositions

$$R \to K(m) \wedge R \xrightarrow{x\times} K(m) \wedge R \qquad\qquad R \to K(m) \wedge R \xrightarrow{\times x} K(m) \wedge R$$

agree. If $m \neq n$, this is obvious since $x$ has image 0 in $K(m)_*(R)$. For $m = n$, it suffices to prove the left and right multiplication by $v_n^a$ induce the same self map of $K(m) \wedge R$. But this is clear, since $v_n$ is the in the image of the unit map $\pi_*(MU_{(p)}) \to \pi_*(K(n))$, hence central. □

**Lemma 11.4.17.** *Let $R$ be a finite $p$-local ring spectrum and $x, y \in \pi_*(R)$ two $v_n$-elements. Then $x^a = y^b$ for some $a, b$.*

*Proof.* By lemma 11.4.15 and lemma 11.4.16, we may assume $x$ and $y$ satisfies $x, y \mapsto 0 \in K(m)_*(R)$ for $m \neq n$ and $x, y \mapsto v_n^j \in K(n)_*(R)$. Moreover, $x$ and $y$ commutes. Since $R$ is of type $> 0$, $x - y$ is again torsion. Finally, by the above construction, $x - y = 0 \in K(m)_*(R)$ for all $m$. Hence $x$ and $y$ satisfies the conditions of lemma 11.4.14, hence $x^{p^j} = y^{p^j}$. Counting in the powers we raised in the above process, we obtain the results. □

**Corollary 11.4.18.** *Let $X$ and $Y$ be spectra which admit $v_n$-self maps $f : \Sigma^a X \to X$ and $g : \Sigma^b Y \to Y$. Let $h : X \to Y$ be any map. Then by replacing $f$ and $g$ by suitable powers, we may assume $a = b$, and the diagram*

$$\begin{tikzcd} \Sigma^a X \arrow[r, "h"] \arrow[d, "f"'] & \Sigma^a Y \arrow[d, "g"] \\ X \arrow[r, "h"'] & Y \end{tikzcd}$$

*commutes*

*Proof.* We view $h$ as a map $e : S \to DX \wedge Y$. Then the commutativity of the diagram is equivalent to $(Df \wedge \mathrm{id}_Y) \circ e \simeq (\mathrm{id}_{DX} \wedge g) \circ e$. This corresponds to two $v_n$ elements in $DX \wedge Y$, hence follows from lemma 11.4.17. □

**Lemma 11.4.19.** *Let $\mathcal{T}$ denote the collection of finite $p$-local spectra which admit a $v_n$-self map. Then $\mathcal{T}$ is closed.*

*Proof.* Clearly $0 \in \mathcal{T}$ and $\mathcal{T}$ is closed under suspension and desuspension. We first show that $\mathcal{T}$ is closed under the formation of cofibers. Let $h : X \to Y$ be a map of finite $p$-local spectra which admits $v_n$-self maps $f$ and $g$. Then by corollary 11.4.18, we may assume that the diagram

$$\begin{tikzcd} \Sigma^k X \arrow[r, "h"] \arrow[d, "f"'] & \Sigma^k Y \arrow[d, "g"] \\ X \arrow[r, "h"'] & Y \end{tikzcd}$$

commutes up to homotopy. Further, by lemma 11.4.15, we may assume $f$ and $g$ are zero maps on $K(m)$-homology for $m \neq n$. Then we consider the map $x : \Sigma^k(Y/X) \to Y/X$ given by the above diagram. We claim that $x$ is a $v_n$-self map. By the long exact sequence of $K(n)$-homology,

$x$ induces an isomorphism on $K(n)$-homology. For $m \neq n$, we consider the diagram of exact sequences

$$\begin{tikzcd} K(m)_{*-k}(Y) \arrow[r] \arrow[d, "0"'] & K(m)_{*-k}(Y/X) \arrow[r] \arrow[d, "x"'] & K(m)_{*-k-1}(X) \arrow[d, "0"] \\ K(m)_*(Y) \arrow[r, "\varphi"] & K(m)_*(Y/X) \arrow[r] & K(m)_{*-1}(X) \end{tikzcd}$$

Then $x$ carries $K(m)_*(Y/X)$ to the image of $\varphi$, and annihilates the image of $\varphi$. Thus $x^2$ is trivial, hence $x$ is a $v_n$-self map.
Finally we prove $\mathcal{T}$ is closed under retracts. For a retract diagram $X \to Y \to X$ and a $v_n$-self map $f : \Sigma^k Y \to Y$, the map $\Sigma^k X \to \Sigma^k Y \to Y \to X$ is a $v_n$-self map of $X$. □

**Theorem 11.4.20.** *There is one finite $p$-local spectrum of type $n$ that admits a $v_n$-self map.*

*Proof.* Omitted. □

*Proof.* theorem 11.4.13 is now obvious. Since $\mathcal{T}$ in lemma 11.4.19, by theorem 11.4.9, coincides with some $\mathcal{C}_{\geq m}$, by theorem 11.4.20 and lemma 11.4.11, implies that $\mathcal{T} = \mathcal{C}_{\geq n}$. □

## 11.5 Localizations

In this section, all localization functors (smashing or not) is assumed to be in the category of $p$-local spectra for some $p$.

**Definition 11.5.1.** Let $X$ be a finite $p$-local spectrum of type $\geq n$. By theorem 11.4.13, we see that $X$ admits a $v_n$-self map, say $f : \Sigma^k X \to X$. Moreover, any two $v_n$-self maps of $X$ coincide after raising to a suitable power. Thus it makes sense to define $X[f^{-1}]$ to be the colimit

$$\varinjlim(X \xrightarrow{f} \Sigma^{-k} X \xrightarrow{f} \Sigma^{-2k} \to \cdots)$$

This is independent of the choice of $f$.

**Proposition 11.5.2.** *By theorem 6.1.2, the thick subcategory $\mathcal{C}_{\geq n+1}$ (the subcategory of finite $p$-local spectrums of type $> n$) generates a subcategory of $p$-local spectra $\mathcal{D}_{\geq n+1}$ under colimits, hence determines a localization functor from the category of $p$-local spectra $\mathcal{D}$ to $\mathcal{D}_{\geq n+1}$. Then, for a finite $p$-local spectrum of type $\geq n$, we have the canonical map $u : X \to X[f^{-1}]$ exhibiting $X[f^{-1}]$ a $\mathcal{D}_{\geq n+1}$-localization of $X$. We denote this localization $L_n^f$. In short, $X[f^{-1}] \simeq L_n^f(X)$.*

*Proof.* By construction 6.1.7, we only need to check two things, namely, the fiber of $u : X \to X[f^{-1}]$ is in $\mathcal{D}_{\geq n+1}$, and $X[f^{-1}]$ is $\mathcal{D}_{\geq n+1}$ local. Notice that $\mathcal{C}_{\geq n+1}$ is itself closed under finite colimits, hence it equivalently generates $\mathcal{D}_{\geq n+1}$ under filtered colimits. Thus $\mathrm{fib}(u) \in \mathcal{D}_{\geq n+1}$ iff $\mathrm{fib}(u)$ is a filtered colimit of objects in $\mathcal{C}_{\geq n+1}$. Also, since $\mathcal{C}_{\geq n+1}$ generates $\mathcal{D}_{\geq n+1}$ under filtered colimits, for $X$ in $\mathcal{D}_{\geq n+1}$ we write $X = \varinjlim X_\alpha$, then $Y$ is $\mathcal{D}_{\geq n+1}$-local iff $[X, Y]_* = 0$ for all $X$, iff $\varinjlim [X_\alpha, Y]_* = 0$ for all $X_\alpha$, and iff $Y$ is $\mathcal{C}_{\geq n+1}$-local.
We first check $\mathrm{fib}(u)$ is a filtered colimit of objects in $\mathcal{C}_{\geq n+1}$. This is since the forming of cofibers commute with filtered colimits, hence we have

$$\mathrm{cof}(u) \simeq \varinjlim(0 \to (\Sigma^{-k} X)/X \to (\Sigma^{-2k} X)/X \to \cdots)$$

Noticce that $(\Sigma^{-nk} X)/X$ is the cofiber of a power of the $v_n$-self map, hence has type $> n$ by construction 11.4.12. Thus the fiber, as a shift of the cofiber, is the filtered colimit of objects in $\mathcal{C}_{\geq n+1}$.
We then check $X[f^{-1}]$ is $\mathcal{C}_{\geq n+1}$-local. Let $Y$ be a spectrum of type $> n$, then to show every map $Y \to X[f^{-1}]$ is null, it suffices to prove $DY \wedge X[f^{-1}]$ is null-homotopic. By lemma 11.4.15, we

may assume $f$ induces the zero map on $K(m)_*(X[f^{-1}])$ for all $m \neq n$. Thus by corollary 11.2.6, we have $K(m)_*(DY \wedge X[f^{-1}]) = 0$ for all $m \neq n$. Thus it now suffices to prove $K(n)_*(DY \wedge X[f^{-1}]) = 0$. But since $Y$ has type $> n$, this is also zero. $\square$

**Corollary 11.5.3.** *The localization $X[f^{-1}]$ or $L_n^f(X)$ is smashing.*

*Proof.* By proposition 11.4.5. $\square$

**Proposition 11.5.4.** *For a finite $p$-local spectrum $X$, we have a fiber sequence*

$$\varinjlim_{k_0,\ldots,k_n} \Sigma^{-n-1} X/(v_0^{k_0}, \ldots, v_n^{k_n}) \to X \to L_n^f(X)$$

*where $v_n$ denote any $v_n$-self map $\Sigma^{\alpha_n} X \to X$.*

*Proof.* We prove by induction. First, for the case $n = 0$, we have $L_0^t(X) \simeq X[p^{-1}]$. Therefore we may form the cofiber of diagrams

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & X \arrow[r, "\mathrm{id}"] \arrow[d] & X \arrow[r] \arrow[d] & \cdots \\
X \arrow[r, "p"] \arrow[d] & X \arrow[r, "p"] \arrow[d] & X \arrow[r] \arrow[d] & \cdots \\
* \arrow[r] & X/p \arrow[r] & X/p^2 \arrow[r] & \cdots
\end{tikzcd}$$

Thus by taking colimits and shifting the resulting fiber sequence of colimits, we have

$$\varinjlim_k \Sigma^{-1} X/p^k \to X \to L_0^t(X)$$

a fiber sequence.

Then, suppose we have proven the case $n$. For $n+1$, we form the following diagram

$$\begin{tikzcd}
\varinjlim_{k_0,\ldots,k_n} \Sigma^{-n-1} X/(v_0^{k_0}, \ldots, v_n^{k_n}) \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & L_n^f(X) \arrow[d] \\
L_{n+1}^t \varinjlim_{k_0,\ldots,k_n} \Sigma^{-n-1} X/(v_0^{k_0}, \ldots, v_n^{k_n}) \arrow[r] & L_{n+1}^t(X) \arrow[r] & L_{n+1}^t L_n^f(X)
\end{tikzcd}$$

Since $\mathcal{C}_{\geq n+1} \supset \mathcal{C}_{\geq n+2}$, the right map is an equivalence. Then, since $L_n^f$ is smashing, it commutes with colimits. Thus the fiber of the left map and the middle map coincide. Thus

$$\mathrm{fib}(X \to L_{n+1}^t(X)) \simeq \varinjlim_{k_0,\ldots,k_n} \mathrm{fib}(\Sigma^{-n-1} X/(v_0^{k_0}, \ldots, v_n^{k_n}) \to L_{n+1}^t \Sigma^{-n-1} X/(v_0^{k_0}, \ldots, v_n^{k_n}))$$

Then, by definition, $X/(v_0^{k_0}, \ldots, v_n^{k_n})$ is of type $> n$, hence $L_{n+1}^t \Sigma^{-n-1} X/(v_0^{k_0}, \ldots, v_n^{k_n})$ is given by $\Sigma^{-n-1} X/(v_0^{k_0}, \ldots, v_n^{k_n})[v_{n+1}^{-1}]$. Therefore the fiber can be identified with

$$\varinjlim_{k_0,\ldots,k_{n+1}} \Sigma^{-n-2} X/(v_0^{k_0}, \ldots, v_{n+1}^{k_{n+1}})$$

$\square$

**Construction 11.5.5.** Notice $\mathcal{C}_{\geq n+1}^{\perp}$ is closed under colimits (as in the proof of proposition 11.5.2). Then we consider the Bousfield localization functor induced by $\mathcal{C}_{\geq n+1}^{\perp}$. We define the colocalization functor to be $D(X)$ and the localization functor $R(X)$. Then the fiber sequence $D(X) \to X \to R(X)$ coincides with the Finish the localization w.r.t $\mathcal{C}_{\geq n+1}^{\perp}$.

Then we prove the following theorem characterizing smashing localizations.

**Theorem 11.5.6.** *Let $L$ be a smashing localization functor. Then $L$ satisfies one of the following conditions.*

1. *$L$ is the zero functor.*
2. *$L$ is the identity functor.*
3. $\ker L_n^f \subseteq \ker L \subseteq \ker L_{E(n)}$.

**Lemma 11.5.7.** *Let $L$ be a localization functor. For $0 \leq n \leq \infty$, we either have $LK(n) \simeq *$ or $LK(n) \simeq K(n)$.*

*Proof.* By the definition of localization, we have a functor $K(n) \to LK(n)$, which gives $LK(n)$ a structure of a $K(n)$-module. Moreover, by corollary 11.3.2, if $LK(n) \neq *$ we have $K(n)$ a retract of $LK(n)$. Thus since the retract of a local object is still local, we have $K(n)$ is local, hence $LK(n) \simeq K(n)$. □

**Lemma 11.5.8.** *Let $L$ be a smashing localization functor and let $E$ be a nonzero complex-oriented cohomology theory whose formal group has height exactly $n$, then $LE \simeq *$ iff $LK(n) \simeq *$.*

*Proof.* If $LE \simeq *$, then $* \simeq K(n) \wedge LE \simeq LK(n) \wedge E$. Since $K(n) \wedge E \neq 0$, $LK(n)$ must be trivial by lemma 11.5.7. □

**Lemma 11.5.9.** *Let $L$ be a smashing localization functor. If $LK(m) \simeq *$, then $LK(n) \simeq *$ for $n > m$.*

*Proof.* For $k \geq 0$, recall $M(k)$ is the cofiber of the map $t_k : \Sigma^{2k} MU_{(p)} \to MU_{(p)}$, and let $R$ be the following ring spectrum

$$R := \bigwedge_{k \neq p^m-1, p^n-1} {}_{MU_{(p)}} M(k) \wedge_{MU_{(p)}} MU_{(p)}[v_n^{-1}]$$

We assume $0 < m < n < \infty$, so that we can write $\pi_*(R) \simeq \mathbb{F}_p[v_m, v_n^{\pm 1}]$. The other cases are essentially the same. Notice $R[v_m^{-1}]$ is a ring spectrum whose formal group law has height exactly $m$. Since $L$ is smashing, it commutes with colimits, hence $LR[v_m^{-1}]$ is the following colimit.

$$\varinjlim (LR \xrightarrow{v_m} \Sigma^{-2(p^m-1)} LR \to \cdots)$$

But by lemma 11.5.8, $LR[v_m^{-1}] \simeq *$. Thus $1 \in \pi_0 LR$ vanishes in $\pi_{2k(p^m-1)}(LR)$ for $k \gg 0$. Or equivalently, $v_m$ is nilpotent in $\pi_*(LR)$, say $v_m^k = 0$. Let $R'$ denote the cofiber of $v_m^{k+1} : \Sigma^{2(k+1)(p^m-1)} R \to R$, hence $v_m^k$ vanishes in $\pi_*(LR')$ (by the long exact sequence, and notice that the homotopy groups of $R$ is concentrated on even degrees). Thus the map on homotopy groups $\pi_*(R') \to \pi_*(LR')$ is not surjective, hence $R'$ is not local. But $R'$ can be obtained as a successive extension of $k+1$ copies of $R/v_m \simeq K(n)$, thus $K(n)$ is not local. By lemma 11.5.7. □

**Lemma 11.5.10.** *Let $L$ be a smashing localization, and $LK(n) \simeq K(n)$. Then every spectrum belonging to $\ker L$ is $E(n)$-acyclic.*

*Proof.* Recall $E(n)$ is Bousfield equivalent to $K(0) \vee \cdots \vee K(n)$. And by lemma 11.5.9, we have $LK(m) \simeq K(m)$ for all $m \leq n$. Finally, we have $K(m) \wedge X \simeq LK(m) \wedge X \simeq K(m) \wedge LX \simeq 0$ for $X \in \ker L$. □

Unfortunately, we will need a theorem that we will prove later (in fact, we still need to prove that the localization w.r.t $E(n)$ is smashing) to analyze the structure of the smashing localization. Namely, we will need theorem 12.2.1.

**Lemma 11.5.11.** *Let $L$ be a smashing localization such that $LK(n) \simeq K(n)$ for $0 \leq n < \infty$. Then $L$ is equivalent to the identity functor.*

*Proof.* By theorem 12.2.1, we have

$$\varprojlim(\cdots \to L_{E(1)}S_{(p)} \to L_{E(0)}S_{(p)}) \simeq S_{(p)}$$

Then, by lemma 11.5.10, we have $L_{E(n)}S_{(p)}$ is $E(n)$-local, hence is local w.r.t $L$. Thus $S_{(p)}$, a limit of $L$-local spectra, is $L$-local. Thus $LX \simeq X \wedge LS_{(p)} \simeq X \wedge S_{(p)} \simeq X$. □

**Theorem 11.5.12.** *Let $L$ be a smashing localization functor, and let $n \geq 0$ be an integer. The following conditions are equivalent.*

1. *$LK(n) \simeq *$.*
2. *$LK(m) \simeq *$ for all $n \leq m \leq \infty$.*
3. *Every finite $p$-local spectrum $X$ of type $\geq n$ belongs to $\ker L$.*
4. *There is one finite $p$-local spectrum $X$ of type $\geq n$ belonging to $\ker L$.*

*Proof.* (1) $\implies$ (2) and (3) $\implies$ (4): Obvious.
(4) $\implies$ (1): Notice $LX \simeq *$ implies $LX \wedge K(n) \simeq X \wedge LK(n) \simeq *$, but since $X$ is of type $n$, $X \wedge K(n) \neq *$, thus $LK(n) \simeq *$, by lemma 11.5.7.
(2) $\implies$ (3): Let $X$ be a finite $p$-local spectrum of type $\geq n$. We prove $LX \simeq *$. Then let $R = X \wedge DX$. Since $LX$ is a $LR$-module, it suffices to prove $LR \simeq *$. Since $LR$ is a ring spectrum, by theorem 11.3.15, it suffices to prove $LR \wedge K(m) \simeq *$ for all $m$. If $m < n$, then obviously $LR \wedge K(m) \simeq L(R \wedge K(m)) = 0$, and $R$ is of type $n > m$. For $m \geq n$, this is obvious by the assumption. □

*Proof.* We now prove theorem 11.5.6.
By theorem 11.5.12, we see that every smashing localization functor is of one of the three cases.

1. $LK(m) \simeq *$ for all $0 \leq m < \infty$.
2. $LK(m) \simeq K(m)$ for all $0 \leq m < \infty$.
3. There is some $n \geq 0$ such that $LK(n) \simeq K(n)$ but $LK(n+1) = 0$.

For the first case, $L$ is the zero functor by lemma 11.5.8. For the second case, this follows from lemma 11.5.11. For the third case, theorem 11.5.12 guarantees that $\ker L$ contains every finite spectrum of type $> n$. Conversely, if $X$ is a finite $p$-local spectrum such that $LX \simeq 0$, we have

$$0 \simeq K(n) \wedge LX \simeq LK(n) \wedge X \simeq K(n) \wedge X$$

hence $X$ has type $> n$. Therefore, we have finite $p$-local spectra belong to $\ker L$ iff it is of type $> n$, that is, iff it is $E(n)$-acyclic. Then by lemma 11.5.10, we have $\ker L_n^f \subseteq \ker L \subseteq \ker L_{E(n)}$. □

*Remark* 20. The last case may seem useless, so Ravenel came up with the telescope conjecture, stating that $L_n^f \simeq L_{E(n)}$. However, this conjecture was disproved in October 2023, by methods of K-theory.

# Chapter 12

# The Chromatic Spectral Sequence

## 12.1 Pro-objects and spectral sequences

**Construction 12.1.1.** Recall for a spectrum $X$ and a ring spectrum $E$, we have an associated cosimplicial ring spectrum $[n] \mapsto X \wedge E^{\wedge n+1}$, and we have the Adams tower of $X$ converging to $\operatorname{Tot} X^\bullet$. That is, we have

$$\operatorname{Tot} X^\bullet \simeq \varprojlim(\cdots \to \operatorname{Tot}_2 X^\bullet \to \operatorname{Tot}_1 X^\bullet \to \operatorname{Tot}_0 X^\bullet)$$

We have an obvious map $X \to \operatorname{Tot} X^\bullet$. We have seen in the case where $E \simeq H\mathbb{F}_p$ that this map gives the information of the $p$-primary part of the homotopy groups of $X$. In general, we consider how this map is close to a homotopy equivalence.

**Proposition 12.1.2.** *For $E$ a ring spectrum and $X$ be any spectrum, then the canonical map $X \to \operatorname{Tot} X^\bullet$ exhibits $\operatorname{Tot} X^\bullet$ as an $E$-localization of $X$ iff $E \wedge \operatorname{Tot} X^\bullet \simeq \operatorname{Tot}(E \wedge X^\bullet)$. Or equivalently, since smash products commutes with finite limits, iff*

$$E \wedge \varprojlim \operatorname{Tot}_n X^\bullet \simeq \varprojlim E \wedge \operatorname{Tot}_n X^\bullet$$

*Proof.* Firstly, notice that for $X \to Y$ an $E$-equivalence, we have a natural homotopy equivalence $E \wedge X \to E \wedge Y$, hence $X^\bullet$ depends only on $L_E(X)$. Then, since $\operatorname{Tot} X^\bullet$ is a inverse limit of $E$-local spectra ($E$-modules, in particular), $\operatorname{Tot} X^\bullet$ is $E$-local. Thus $\operatorname{Tot} X^\bullet$ is an $E$-localization of $X$ iff $X \to \operatorname{Tot} X^\bullet$ induces an isomorphism on $E$-homology. Equivalently, iff $E \wedge X \to E \wedge \operatorname{Tot} X^\bullet$ is an homotopy equivalence.
Obviously, there are natural maps $\operatorname{Tot} X^\bullet \to \operatorname{Tot}_i X^\bullet$, hence a natural map $E \wedge \operatorname{Tot} X^\bullet \to \operatorname{Tot}(E \wedge X^\bullet)$. However, notice that $E \wedge X^\bullet$ is a split augmented cosimplicial object with the $-1$-th space $E \wedge X$. Thus the composition $E \wedge X \to E \wedge \operatorname{Tot} X^\bullet \to \operatorname{Tot}(E \wedge X^\bullet)$ is a homotopy equivalence. □

**Definition 12.1.3.** Recall that the pro-category is formed by formally adjoining to a category all filtered limits. In particular, we define $\operatorname{Pro}(\mathbf{Sp})$ to be the category with objects $\varprojlim X_\alpha$ and morphisms $\operatorname{Hom}(\varprojlim X_\alpha, \varprojlim Y_\beta) \simeq \varprojlim_\beta \varinjlim_\alpha (X_\alpha, Y_\beta)$ (this notion makes sense since the essential image of the Yoneda embedding is cocompact). We have a natural forgetful functor $U : \operatorname{Pro}(\mathbf{Sp}) \to \mathbf{Sp}$ given by $\varprojlim X_\alpha \mapsto \varprojlim X_\alpha$, where the second limit is taken in $\mathbf{Sp}$.
We say a pro-spectrum $\varprojlim X_\alpha$ is constant if it is equivalent to the constant tower $X$. Or, formally, $\varprojlim X_\alpha \simeq X$.
The monoidal structure on $\mathbf{Sp}$ somewhat extends to $\operatorname{Pro}(\mathbf{Sp})$ by defining $E \wedge \varprojlim X_\alpha = \varprojlim(E \wedge X_\alpha)$. Since the smash product does not necessarily commutes with limits, the map $E \wedge U(\varprojlim X_\alpha) \to U(E \wedge \varprojlim X_\alpha)$ is not always an equivalence, but in particular it is an equivalence when $X_\alpha$ is constant.

**Corollary 12.1.4.** *The equivalent conditions of proposition 12.1.2 is satisfied when the tower*

$$\cdots \to \mathrm{Tot}_2 X^\bullet \to \mathrm{Tot}_1 X^\bullet \to \mathrm{Tot}_0 X^\bullet$$

*is constant as a pro-spectrum.*

So, to be able to apply the corollary, we need a condition to determine if a tower is constant as a pro-spectrum. This is the following criterion given by Bousfield.

**Theorem 12.1.5.** *Suppose we have a tower of spectra*

$$\cdots \to Y(2) \to Y(1) \to Y(0)$$

*Then, suppose there is some $s \geq 1$ such that for every finite spectrum $F$, let $\{E_r^{p,q}, d_r\}$ be the spectral sequence (for a full account see nlab or of the dual sequence see Higher Algebra 1.2.2, but I will give a brief introduction in the proof) associated to the tower*

$$\cdots \to F \wedge Y(2) \to F \wedge Y(1) \to F \wedge Y(0)$$

*then the groups $E_s^{p,q}$ vanish for $p \geq s$, then the tower $Y$ is constant as a pro-spectrum.*

Before the proof of the theorem, we need a technical lemma.

**Lemma 12.1.6.** *The composition of two phantom maps is null.*

*Proof.* Fix a spectrum $X$, and let $u : \bigvee F_\alpha \to X$ be the direct sum of all maps from finite spectra to $X$. Then $X' := \mathrm{fib}(u)$ is the retract of a sum of finite spectra. Then, suppose $f : X \to Y$ and $g : Y \to Z$ be phantom. Since $f$ is phantom, $f \circ u = 0$, and hence $f$ is equivalent to the composition $X \to \Sigma X' \to Y$. Then, $g \circ f$ factors as $X \to \Sigma X' \to Y \to Z$. Since $g$ is phantom and $\Sigma X'$ is a retract of a sum of finite spectra, this map is null. □

*Proof.* We first construct the spectral sequence. For $m \leq n$, we let $F(n,m)$ denote the homotopy fiber of the map $Y(n) \to Y(m)$, and let $E_r^{p,q}$ be defined as the image of the map $\pi_q(F(p+r-1,p-1)) \to \pi_q(F(p,p-r))$, and $\mathrm{d}_r$ carries $E_r^{p,q}$ to $E_r^{p+r,q-1} \subseteq \pi_{q-1}(F(p+r,p))$ induced by the fiber sequence $F(p+r,p) \to F(p+r,p-r) \to F(p,p-r)$. Notice that if $p<0$, then $F(p,p-r)$ is contractible and thus $E_r^{p,q}$ vanishes. If $p \geq s$, then $E_r^{p,q}$ vanishes by assumption. Thus, if $r \geq s$, then at least one of the groups $E_r^{p,q}$ and $E_r^{p+r,q-1}$ vanishes, hence $d_r$ is always zero. Thus we have the spectral sequence collapses at the $s$-th space.
Now, we suppose $r > p$. Since $F(p,p-r) \simeq Y(p)$, we have $\pi_q(F(p,p-r)) \simeq \pi_q(Y(p))$. Also, note that $E_r^{p,q}$ is the image of the composite map

$$\pi_q(F(p+r-1,p-1)) \to \pi_q(Y(p+r-1)) \to \pi_q(Y(p))$$

By definition, the image of the first map is the kernel of $\pi_q(Y(p+r-1)) \to \pi_q(Y(p-1))$. And notice the map $\pi_q(Y(p+r-1)) \to \pi_q(Y(p-1))$ factors through the map $\pi_q(Y(p)) \to \pi_q(Y(p-1))$. It follows that for $r > p$, the group $E_r^{p,q}$ is the intersection $\mathrm{im}(\pi_q(Y(p+r-1)) \to \pi_q(Y(p))) \cap \ker(\pi_q(Y(p)) \to \pi_q(Y(p-1)))$. Thus for $r > p, s$, the intersecion is independent for $r$.
Then, we prove by induction that the intersection $\mathrm{im}(\pi_q(Y(p+r)) \to \pi_q(Y(p))) \cap \ker(\pi_q(Y(p)) \to \pi_q(Y(p-k)))$ is independent of $r \gg 0$ for all $k$. The case $k=0$ is trivial, and we assume $k>0$. Suppose $r > p, s$, and $x \in \pi_q(Y(p+r))$ has trivial image in $\pi_q(Y(p-k))$. Let $y$ be the image of $x$ in $\pi_q(Y(p))$. We wish to show that $y$ lifts to $\pi_q(Y(p+r+1))$. Let $y'$ denote the image of $y$ in $\pi_q(Y(p-1))$. Then $y'$ belongs to the kernel of the map $\pi_q(Y(p-1)) \to \pi_q(Y(p-k))$. Since $y'$ lifts to $\pi_q(Y(p+r))$, the induction hypothesis shows that $y'$ can be lifted to $x' \in \pi_q(Y(p+r+1))$. Subtracting the image of $x'$ from $x$ we may assume $y' = 0$, and thus $y \in \ker(\pi_q(Y(p)) \to \pi_q(Y(p-1)))$, and thus we have proven the result.
Then we take $k = p-1$, we see that the image of the map $\pi_*(Y(p+r)) \to \pi_*(Y(p))$ is independent

of $r$ as long as $r \geq p, s$. We denote this image by $A(p)_*$. By construction, we have a sequence of surjections

$$\cdots \to A(2)_* \to A(1)_* \to A(0)_*$$

Thus by the construction of the spectral sequence, they fit in exact sequences

$$0 \longrightarrow E_\infty^{p,*} \longrightarrow A(p)_* \longrightarrow A(p-1)_* \longrightarrow 0$$

By assumption, $E_\infty^{p,*}$ vanishes for $p \geq s$. Thus $A(p)_* \to A(p')_*$ are isomorphisms for $p \geq p' \geq s$. This encourages us to consider the tower

$$\cdots \to \pi_*(Y(4s)) \xrightarrow{\theta_2} \pi_*(Y(2s)) \xrightarrow{\theta_1} \pi_*(Y(s))$$

For $m \geq 0$, let $Z(m)_* \subseteq \pi_*(Y(2^m s))$ be the kernel of $\theta_m$. Thus $Z(m)_* \cap A(2^m s)_* = 0$, since $\theta_m$ restricts to an isomorphism $A(2^m s)_* \to A(2^{m-1}s)_*$. Then by definition, $\theta_m$ maps onto $A(2^{m-1}s)_*$. Thus we have a direct sum decomposition $\pi_*(Y(2^m s)) \simeq A(2^m s)_* \oplus Z(m)_*$. Thus $\{\pi_*(Y(2^m s))\}$, as a pro-object in graded Abelian groups, is equivalent to the constant group $A(s)_*$.
Let $Y = \varprojlim Y(p) \simeq \varprojlim_m Y(2^m s)$. The Milnor exact sequence

$$0 \longrightarrow \varprojlim^1 \pi_{*+1}(Y(p)) \to \pi_*(Y) \to \varprojlim \pi_* Y(p) \to 0$$

gives $\pi_*(Y) = A(s)_*$. For each integer $p \geq 0$, let $Y(p)/Y$ denote the cofiber of the canonical map $Y \to Y(p)$. Thus by considering the direct sum decomposition, we have the tower

$$\cdots \to Y(4s)/Y \to Y(2s)/Y \to Y(s)/Y$$

has each map in the tower trivial on all homotopy groups. By applying the above argument to $(Y(2^m s)/Y)_* F$ for all finite $F$, we see that each map is a phantom. Now, by lemma 12.1.6, we have the tower

$$\cdots \to Y(16s)/Y \to Y(4s)/Y \to Y(s)/Y$$

consisting of null maps, hence we have this tower equivalent to $*$. Thus $\{Y(p)\}$ is equivalent to $Y$. $\square$

**Corollary 12.1.7.** *Let $E$ be a ring spectrum and $X$ be any spectrum. Suppose that there is an integer $s \geq 1$ such that for every finite spectrum $F$, the $E$-based Adams spectral sequence $\{E_r^{p,q}, \mathrm{d}_r\}$ for $X \wedge F$ has $E_r^{p,q} = 0$ for $p \geq s$. Then the Adams tower $\{\mathrm{Tot}_n X^\bullet\}$ is equivalent as a pro-object to a constant tower. Thus, the Adams tower exhibits $\mathrm{Tot}(X^\bullet)$ as $L_E(X)$. Also, the canonical map $L_E(X) \wedge Y \to L_E(X \wedge Y)$ is an equivalence for any other $Y$.*

*Proof.* Apply theorem 12.1.5 and corollary 12.1.4 to the Adams tower. $\square$

**Lemma 12.1.8.** *Let $E$ be a $p$-local ring spectrum, and suppose that there is one finite $p$-local spectrum $X$ of type $0$ such that the condition in corollary 12.1.7 holds. Then $L_E$ is a smashing localization.*

*Proof.* Let $\mathcal{T}$ be the collection of finite $p$-local spectra such that for every $p$-local spectrum $Y$ the map $L_E(X) \wedge Y \to L_E(X \wedge Y)$ is an equivalence. Clearly $\mathcal{T}$ is thick. Then, since $\mathcal{T}$ contains $X$, which is of type 0, by theorem 11.4.9, we have $\mathcal{T}$ contains every finite $p$-local spectrum. Thus $S_{(p)} \in \mathcal{T}$, and $L_E(Y) \simeq L_E(S_{(p)}) \wedge Y$ for all $Y$, hence $L_E$ is smashing. $\square$

We now prove theorem 12.1.9.

**Theorem 12.1.9.** *The localization $L_{E(n)}$ is smashing.*

**Lemma 12.1.10.**

## 12.2 The chromatic tower

**Theorem 12.2.1.**

## 12.3 The J-homomorphism

# Chapter 13

# Appendix: Formal Schemes

## 13.1 Basic definitions

**Construction 13.1.1.** Recall that we have the category of affine schemes **Aff** dual to the category **Ring**. Therefore, by the Yoneda lemma, we may regard the category **Aff** as the full subcategory of $\mathrm{Fun}(\mathbf{Ring}, \mathbf{Set})$ consisting of representable objects. Obviously $\mathbf{Aff} \simeq \mathbf{Ring}^{\mathrm{op}}$. The following chapter will mainly be focused on affine schemes in this viewpoint.

We begin with some basic affine schemes.

**Example 4.** *For a ring $A$, we define $\operatorname{Spec} A$ to be the functor $B \mapsto \mathrm{Hom}(A, B)$. Clearly, all affine schemes are of this form.*
*We define the scheme $\mathbb{A}^1$ to be the forgetful functor $\mathbf{Ring} \to \mathbf{Set}$. This is representable by the ring $\mathbb{Z}[x]$.*
*We define the scheme $\mathbb{G}_m$ to be the functor $R \mapsto R^*$, the units in $R$. This is representable by the ring $\mathbb{Z}[x, x^{-1}]$.*
*We define the scheme $\mathrm{FGL}$ to be the functor $R \mapsto \mathrm{FGL}(R)$. Recall from definition 9.1.3 that we have $\mathrm{FGL}$ is indeed a scheme.*

Then we imitate the definition of the structure sheaf on a scheme $X$.

**Definition 13.1.2.** For any affine scheme $X$, we define $\mathcal{O}_X$ to be the set $\mathrm{Hom}_{\mathbf{Aff}}(X, \mathbb{A}^1)$. Notice $\mathbb{A}^1$ is a ring object: the scheme $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$ sends a ring $R$ to two elements in $R$, and the structure map is given by the addition and multiplication of the two elements. Then $\mathrm{Hom}_{\mathbf{Aff}}(X, \mathbb{A}^1)$ inherits a natural ring structure.

*Remark* 21. By the identification $\mathbf{Aff} \simeq \mathbf{Ring}^{\mathrm{op}}$, for $X = \operatorname{Spec} A$, we clearly have $\mathrm{Hom}_{\mathbf{Aff}}(X, \mathbb{A}^1) \simeq \mathrm{Hom}_{\mathbf{Ring}}(\mathbb{Z}[x], A) \simeq A$.

**Lemma 13.1.3.** $X(-) \simeq \mathrm{Hom}_{\mathbf{Ring}}(\mathcal{O}_X, -)$.

*Proof.* Suppose $X$ is represented by a ring $R$, then $X(-) \simeq \mathrm{Hom}_{\mathbf{Ring}}(R, -)$. But then, $\mathrm{Hom}_{\mathbf{Aff}}(X, \mathbb{A}^1) \simeq \mathrm{Hom}_{\mathbf{Ring}}(\mathbb{Z}[x], R) \simeq R$. □

**Definition 13.1.4.** For an affine scheme $X$, we define a point of $X$ to be a ring $R$ and an element $x \in X(R)$. For any scheme, we may define its category of points $\mathrm{Pt}(X)$ with object pairs of the form $(R, x)$, and we have a morphism $f : (R, x) \to (S, y)$ if $f : R \to S$ is a morphism of rings and $X(f)(x) = y$. For a point $(R, x)$, we let $\mathcal{O}_x$ denote its underlying ring.

**Definition 13.1.5.** Notice that for $(R, x)$ a point of $X$, and $f \in \mathcal{O}_X$, by definition $f$ is a morphism from $X$ to $\mathbb{A}^1$. Thus $(R, x)$ push forwards to a point of $\mathbb{A}^1$, namely, an element of $R$. We will denote this element by $f(x)$.

**Definition 13.1.6.** For a scheme $X$ and an ideal $I \leq \mathcal{O}_X$, we define a scheme $V(I)$ by

$$V(I)(R) = \{x \in X(R) | f(x) = 0, \forall f \in I\}$$

We call schemes of this form closed subschemes of $X$.
Similarly, for a scheme $X$ and an element $f \in \mathcal{O}_X$, we define a scheme $D(f)$ by

$$D(f)(R) = \{x \in X(R) | f(x) \in R^*\}$$

We call these schemes the basic open subschemes of $X$.
A locally closed subscheme is a basic open subscheme of a closed subscheme, which is of the form $D(f) \cap V(I) = D(f) \times_X V(I)$.
Finally, we have the intersection of subschemes $D(f) \cap D(g) = D(fg)$, $V(I) \cap V(J) = V(I+J)$.
We define the union of closed subschemes to be $V(I) \cup V(J) = V(I \cap J)$, and the schematic union by $V(I) + V(J) = V(IJ)$.

*Remark* 22. The union and the schematic union does not coincide in most cases. The schematic union counts multiplicity by infinitesimal thickening. For example, $V(I) + V(I) = V(I^2)$.

We finally recall some basic definitions in algebraic geometry.

**Definition 13.1.7.** We define a scheme $X$ to be reduced if $\mathcal{O}_X$ has no non-zero nilpotents, and define $X_{red} = \mathrm{Spec}(\mathcal{O}_X/\sqrt{0})$. We have a natural inclusion $X_{red} \hookrightarrow X$ induced by the quotient map $\mathcal{O}_X \to \mathcal{O}_X/\sqrt{0}$.
We define $X$ to be connected if it cannot be split into $Y \amalg Z$, equivalently if there is no non-trivial idempotents in $\mathcal{O}_X$.
We define $X$ to be integral if $\mathcal{O}_X$ is an integral domain, and irreducible if $X_{red}$ is integral.
We define $X$ to be Noetherian if $\mathcal{O}_X$ is Noetherian. Then, $X_{red} = \bigcup Y_i$, where $Y_i$ is integral and closed. This decomposition is unique, with $Y_i$ corresponding to the schemes $V(\mathfrak{p}_i)$, where $\mathfrak{p}_i$ are minimal prime ideals of $\mathcal{O}_X$.

## 13.2 Sheaves

**Definition 13.2.1.** For a functor $X \in \mathrm{Fun}(\mathbf{Ring}, \mathbf{Set})$, we define a sheaf $M$ over $X$ with the following data.

1. For each $(R, x) \in \mathrm{Pt}(X)$, an $R$-module $M_x$.
2. For each map $f : (R, x) \to (S, y) \in \mathrm{Pt}(X)$, an isomorphism $\theta(f) : S \otimes_R M_x \to M_y$, satisfying the natural functorality conditions.

For sheaves $M$ and $N$ over $X$, we define a map $f : M \to N$ to be maps $f_x : M_x \to N_x$ for all points of $x$ and commuting with all the maps $\theta(f)$. With this definition, we denote $\mathbf{Sh}(X)$ for the category of all sheaves over $X$,

**Lemma 13.2.2.** *The category $\mathbf{Sh}(X)$ has direct sums and tensor products. Moreover, the unit of the direct sum is the zero sheaf, with $M_x = 0$ for all $x$. The unit of the tensor product is the sheaf $\mathcal{O}$ defined on each $(R, x)$ to be $R$.*

*Proof.* Direct by using the pointwise definitions. □

Similar to the classical methods in algebraic geometry, we can form a sheaf on a scheme using a module.

**Definition 13.2.3.** Let $X$ be a scheme and $N$ be an $\mathcal{O}_X$-module. Then we may define a sheaf $\tilde{N}$ over $X$ by letting $N_x = R \otimes_{\mathcal{O}_x} N$, where $R$ is given a $\mathcal{O}_X$-algebra structure by $f(r) = f(x)r$.

Also, we may form the vector bundle scheme using a sheaf on $X$. Through this definition, we may define the sections of a sheaf.

**Definition 13.2.4.** For a scheme $X$ and a sheaf $M$ over $X$, we define $\mathbb{A}(M) : \mathbf{Ring} \to \mathbf{Set}$ (not necessarily an affine scheme) as $\mathbb{A}(M)(R) = \bigoplus_{x \in X(R)} M_x$. Notice that we have a natural transformation $\mathbb{A}(M) \to X$ by sending $M_x$ to $x$. Therefore we have $\mathbb{A}(M) \in \mathrm{Fun}(\mathbf{Ring}, \mathbf{Set})/X$. For a scheme $Y$ over $X$ and a sheaf $M$, we define $\Gamma(Y, M) = \mathrm{Hom}_{\mathrm{Fun}(\mathbf{Ring},\mathbf{Set})/X}(Y, \mathbb{A}(M))$.

**Proposition 13.2.5.** *When $X$ is an affine scheme, we have $\Gamma(X, -) : \mathbf{Sh}(X) \to \mathbf{Mod}(\mathcal{O}_X)$ an equivalence of categories.*

*Proof.* We claim $\Gamma(X, M) \simeq M_{x_0}$, where $x_0$ is the canonical element in $X(\mathcal{O}_X)$. Firstly, the isomorphism is given by $u \mapsto$ $\square$

# Chapter 14

# Appendix: Algebraic Stacks

## 14.1 Basic Definitions

As is well-known, stacks are generalizations to sheaves, similar to groupoids generalizing sets. Namely, a stack is a sheaf taking values in groupoids. We give the formal definition as follows (we will focus only on the theory of stacks here, but the arguments apply easily to $\infty$-stacks).

**Definition 14.1.1.** Let $\mathcal{C}$ be a category, we define a sieve on $\mathcal{C}$ to be a full subcategory $\mathcal{C}'$ such that if $f : c \to d$ is a morphism in $\mathcal{C}$, and $d \in \mathcal{C}'$, then we have $c \in \mathcal{C}'$. For $c \in \mathcal{C}$ an object, we define a sieve on $c$ to be a sieve on the over category $\mathcal{C}_{/c}$.
Notice that if we have a functor between categories $F : \mathcal{C} \to \mathcal{D}$, then for a sieve $\mathcal{D}'$ on $\mathcal{D}$, the category $\mathcal{D}' \times_{\mathcal{D}} \mathcal{C}$ is a sieve on $\mathcal{C}$. We denote this sieve by $F^*\mathcal{D}'$. This construction applies to a morphism $f : c \to d$ and a sieve $S$ on $d$, since $f$ induces a functor $\mathcal{C}_{/c} \to \mathcal{C}_{/d}$.

**Definition 14.1.2.** A category $\mathcal{C}$ equipped with sieves on objects (called covering sieves) is called a site if the three following properties are satisfied.

1. For $c \in \mathcal{C}$, $\mathcal{C}_{/c}$ is a covering sieve.
2. For $c \in \mathcal{C}$ and $S$ a covering sieve on $c$, then for any $f : d \to c$, $f^*S$ is a covering sieve on $d$.
3. For $c \in \mathcal{C}$, $S$ a covering sieve on $c$, and $T$ an arbitrary sieve on $c$, if for all $f : d \to c$ in $S$ we have $f^*T$ a covering sieve on $d$, then $T$ is a covering sieve on $c$.

**Construction 14.1.3.** For a sieve $S$ on $c$, we define a subobject ($(-1)$-truncated morphism) of $j(c)$ (here $j$ is the Yoneda embedding) $j(S)$ as follows. We first recall that for a object $c \in \mathcal{C}$, the functor $F : \mathcal{P}(\mathcal{C}_{/c}) \to \mathcal{P}(\mathcal{C})_{/j(c)}$ extending $\mathcal{C}_{/c} \to \mathcal{P}(\mathcal{C})_{/j(c)}$ is an equivalence of categories. Then, a sieve $S$ on $c$ induces a functor $\mathcal{C}_{/c} \to \Delta^1$, where the elements in $S$ are sent to 0 and others to 1. We consider the inclusion $\Delta^1 \hookrightarrow \infty\mathbf{Grpd}^{\mathrm{op}}$ given by $0 \mapsto *$ and $1 \mapsto \phi$. Then we obtain a presheaf $\mathcal{P}(\mathcal{C}_{/c})$. By applying $F$ to this element, we have an element in $\mathcal{P}(\mathcal{C})_{/j(c)}$. Also, notice that $\Delta^1$ is the $(-1)$-truncated object classifyer, the element we obtained is a $(-1)$-truncated morphism $j(S) \hookrightarrow j(c)$.

**Definition 14.1.4.** We call a functor (presheaf) $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \infty\mathbf{Grpd}$ a stack if we have for all $c$ and all sieves $S$ on $c$, we have $\mathrm{Hom}_{\mathcal{P}(\mathcal{C})}(j(c), \mathcal{F}) \to \mathrm{Hom}_{\mathcal{P}(\mathcal{C})}(j(S), \mathcal{F})$ is an isomorphism.

However, by the straightening and unstraightening theorem, we have another perspective on stacks.

**Definition 14.1.5.** We define a functor $F : \mathcal{F} \to \mathcal{C}$ to be a left fibration if it has the right lifting properties with respect to inclusions $\Lambda^n_i \hookrightarrow \Delta^n$, where $0 \le i < n$. This implies the fiber of $c \in \mathcal{C}$ is a groupoid, which we will denote by $\mathcal{F}(c)$.

**Construction 14.1.6.** Let $\mathcal{F} \to \mathcal{C}$ be a left fibration. Then let $S$ be a covering sieve on $c$ (or equivalently, let $c_i \to c$ be a family of morphisms generating the covering sieve), then we define the category of descent data $\mathrm{Desc}(S, \mathcal{F})$. This category has objects as systems $(X_i, \theta_{i,j})$, where $X_i$ are objects of $\mathcal{F}$ over $c_i$, and $\theta_{i,j} : X_i|_{c_{i,j}} \to X_j|_{c_{i,j}}$ are isomorphisms. The maps $\theta_{i,j}$ are required to satisfy the cocycle conditions. The morphisms from $(X, \theta)$ to $(Y, \rho)$ are families of arrows $f_i : X_i \to Y_i$ over $c_i$ such that $\rho_{i,j} f_j = f_i \theta_{i,j}$.
Then, notice we have an obvious functor $\mathcal{F}(c) \to \mathrm{Desc}(S, \mathcal{F})$ given by restriction.

**Definition 14.1.7.** We define a stack to be a left fibration $F : \mathcal{F} \to \mathcal{C}$ such that the functor $\mathcal{F}(c) \to \mathrm{Desc}(S, \mathcal{F})$ is an equivalence for all $c$ and all covering sieves on $c$.

**Theorem 14.1.8.** *The two definitions of stacks are equivalent.*

*Proof.* The equivalence is given by the straightening and unstraightening construction, and we see that the condition $\mathcal{F}(c) \simeq \mathrm{Desc}(S, \mathcal{F})$ is equivalent to the locality (on sieves) condition. □

Although the writer of this note prefers the first definition, we will adapt the second due to historical reasons. We begin with a theorem determining whether a left fibration is a stack.

**Proposition 14.1.9.** *The above definition is equivalent to the two following conditions.*

1. *For all $X$, $Y$ over $c$, the functor $\mathrm{Iso}_c(X, Y) : \mathcal{C}_{/c} \to \mathbf{Set}$ sending $f : c' \to c$ to the set of isomorphisms between $f^*X$ and $f^*Y$ is a sheaf.*
2. *For a covering $c_i \to c$, and a collection $X_i$ over $c_i$ with isomorphisms $\theta_{i,j} : X_i|_{c_{i,j}} \to X_j|_{c_{i,j}}$, there is a $X$ over $c$ with $X|_{c_i} \simeq X_i$ inducing the above isomorphisms.*

*Proof.* One condition corresponds to the fully faithfulness of the functor, and the other essentially surjectiveness. □

**Definition 14.1.10.** For stacks $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ over a category $\mathcal{C}$ with structure morphism $p_\mathcal{F}$, $p_\mathcal{G}$, and $p_\mathcal{H}$, we define a morphism between $\mathcal{F}$ and $\mathcal{G}$ to be a functor $f : \mathcal{F} \to \mathcal{G}$ such that $p_\mathcal{G} \circ f = p_\mathcal{F}$. We define $f \circ g = h : \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ if there is a natural isomorphism between the two functors. We define the category $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ to have objects morphisms, and morphisms natural transformations.

From now on, we restrict our discussion to algebraic geometry, in which $\mathcal{C}$ is always the category of schemes or the category of schemes over $S$, equipped with some topology (for example Zariski, etale, fppf).

**Definition 14.1.11.** For a scheme $X \in \mathbf{Sch}_{/S}$, we define the stack associated to $X$ to be the left fibration $p_X : \mathbf{Sch}_{/X} \to \mathbf{Sch}_{/S}$, where $p_X$ is the natural projection. Similarly we may define a stack associated to an algebraic space (this is a special case of a stack where the preimage of $A \in \mathbf{Sch}_{/S}$ is discrete). In this case, we say the stack is representable by a scheme or algebraic space.

**Construction 14.1.12.** The fiber product of stacks are given by homotopy pullbacks of categories (or, as is more well-known, the comma category).

**Definition 14.1.13.** We define a morphism of stacks $f : \mathcal{F} \to \mathcal{G}$ is representable if for all $U \in \mathbf{Sch}$ and morphisms $U \to \mathcal{G}$ (note here $U$ denote the stack associated to $U$, and we will use this notation repeatedly), the fiber product $U \times_\mathcal{G} \mathcal{F}$ is representable by an algebraic space.
Moreover, for a property $\mathfrak{P}$ of morphisms of schemes, if $\mathfrak{P}$ is local w.r.t our chosen topology on $\mathbf{Sch}$ and is stable under basechange (for example separated, quasi-compact), then we say a morphism of stacks $f : \mathcal{F} \to \mathcal{G}$ is $\mathfrak{P}$ iff for all $U$ and morphisms $U \to \mathcal{G}$, the morphism $U \times_\mathcal{G} \mathcal{F}$ is $\mathfrak{P}$.

**Definition 14.1.14.** We define a Deligne-Mumford (Artin) stack to be a stack $\mathcal{F}$ on $\mathbf{Sch}/S$ with the etale (fppf) topology, satisfying the two following properties.

1. The diagonal $\mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$ is representable, quasi-compact and separated.

2. There is a scheme $U$ (called an atlas, and has the intuition of an atlas) and a etale surjective morphism $U \to \mathcal{F}$.

We finish this section with an important example (especially important for this notes).

**Example 5.** *For a scheme $X$ over $S$ and an affine flat group $G$ over $S$, we define the stack $[X/G]$ as follows. This stack has the objects over $B \in \mathbf{Sch}/S$ defined as principal $G$-bundles $\pi : E \to B$ equipped with a $G$-equivariant map $f : E \to X$. Morphisms over $g : B \to B'$ are commutative diagrams*

$$\begin{tikzcd} E \arrow[r] \arrow[d] \arrow[rr, bend left] & E' \arrow[r] \arrow[d] & X \\ B \arrow[r] & B' & \end{tikzcd}$$

*Firstly, it is a stack by the criterion proposition 14.1.9: $Iso_B(e, e')$ is non-empty iff $e \simeq e'$ as principal bundles, and when it is non-empty, it is classified by the stablizer of the identity map $X \to X$, hence is a sheaf represented by a scheme. The second condition follows by the fppf descent of principal bundles. Then, we claim it is an Artin stack. Firstly, for any map $U \to [X/G] \times_S [X/G]$ (equivalent to the data of two objects $Y_1$, $Y_2$ over $U$), notice that the pullback $U \times_{[X/G]\times_S[X/G]} [X/G] \to U$ is equivalent to the morphism $\mathrm{Iso}_U(Y_1, Y_2) \to U$. However recall by the above construction that* Iso *finish the proof!*

## 14.2 Constructions

**Definition 14.2.1.** For a stack $\mathcal{F}$, we define a stack $\mathcal{G}$ to be a substack of $\mathcal{F}$ if it is a full subcategory of $\mathcal{F}$, and

1. If $X \in \mathcal{F}$ is in $\mathcal{G}$, then isomorphic objects are also in $\mathcal{G}$.

2. For all morphism of schemes $f : U \to V$, if $X$ over $V$ is in $\mathcal{G}$, then so is $f^*X$.

3. Let $U_i \to U$ be a cover, then $X$ over $U$ is in $\mathcal{G}$ iff $X|_{U_i}$ is in $\mathcal{G}$ for all $i$.

**Definition 14.2.2.** We define $\mathcal{G}$ to be an open (closed) substack of $\mathcal{F}$ if it is a substack and the inclusion functor $\mathcal{G} \to \mathcal{F}$ is representable and an open (closed) immersion.

**Definition 14.2.3.** For an algebraic stack $\mathcal{F}$ over $S$, the set of points of $\mathcal{F}$ is defined to be the set of equivalence classes of pairs $(k, x)$, where $k$ is a field over $S$, and $x : \operatorname{Spec} k \to \mathcal{F}$ a morphism of stacks over $S$. Two pairs $(k, x)$ and $(k', x')$ are equivalent if there is a common field extension $k''$ and a map $x'' : \operatorname{Spec} k'' \to \mathcal{F}$ extending $x$ and $x'$. Clearly every point of an algebraic stack is an image of a point of its atlas.

**Definition 14.2.4.** For an algebraic stack $\mathcal{F}$ with atlas $u : X \to \mathcal{F}$, and $\xi$ a point of $\mathcal{F}$ coming from $x \in X$, we then define the dimension of $\mathcal{F}$ at $\xi$ to be

$$\dim_\xi \mathcal{F} = \dim_x X - \dim_x f$$

where $\dim_x f$ is defined to be the relative dimension of the pullback $f^{-1}(\xi) \to \xi$. We define the dimension of the algebraic stack to be the maximum dimension of its points.

*Remark* 23. The dimension may be negative.

**Definition 14.2.5.** We define a quasi-coherent sheaf $\mathcal{S}$ on an algebraic stack $\mathcal{F}$ to consist of the following data. For each morphism from a scheme $X \to \mathcal{F}$, we have a quasi-coherent sheaf $\mathcal{S}_X$ on $X$, and for each $f : X \to Y$ commuting with the morphisms to $\mathcal{F}$, we have an isomorphism $\varphi_f : \mathcal{S}_X \to f^*\mathcal{S}_Y$. Moreover, for morphisms $g \circ f : X \to Y \to Z$, we must have $(f^*\varphi_g) \circ \varphi_f = \varphi_{g\circ f}$.
We say $\mathcal{S}$ is coherent (flat, faithfully flat, finite type, finite presentation, locally free) if $\mathcal{S}_X$ satisfy the corresponding property for all $X$.
The morphisms are defined obviously by $h_X : \mathcal{S}_X \to \mathcal{S}'_X$ commuting with the isomophisms $\varphi$.

**Example 6.** *We define the structure sheaf $\mathcal{O}_{\mathcal{F}}$ of a algebraic stack $\mathcal{F}$ to be $(\mathcal{O}_{\mathcal{F}})_X = \mathcal{O}_X$.*

**Theorem 14.2.6.** *The category of quasi-coherent sheaves over an algebraic stack $\mathcal{F}$, denoted by* **QCoh**$(\mathcal{F})$*, is an Abelian category.*

*Proof.* The proof is similar to the fact that the quasi-coherent sheaves over a scheme is an Abelian category. □

# Chapter 15

# Appendix: Deformation Theory

# Part IV

# K-theory

# Chapter 16

# Basic constructions in K-theory

## 16.1 Group completion

**Definition 16.1.1.** In a category $\mathcal{C}$ viewed as a Cartesian symmetric monoidal category, we define $\mathbb{E}_n\mathrm{Mon}(\mathcal{C})$ to be the category of $\mathbb{E}_n$-monoids in $\mathcal{C}$. In particular when $n = 1$ or $n = \infty$, we denote the category by $\mathrm{Mon}(\mathcal{C})$ and $\mathrm{CMon}(\mathcal{C})$.
We say a monoid $X$ to be grouplike if the shear map $(\pi_1, \mu) : X \times X \to X \times X$ is an equivalence, where $\pi_1$ is the projection to the first factor, and $\mu$ is the multiplication map. We denote the category of grouplike $\mathbb{E}_n$-monoids to be $\mathbb{E}_n\mathrm{Grp}(\mathcal{C})$ (the notation for $n = 1$ or $n = \infty$ is still adopted).
Notice that any monoid of $\mathcal{C}$ is naturally pointed by the unit map.

**Theorem 16.1.2.** *We have equivalences of categories*

$$\mathrm{Grp}(\mathcal{C}) \simeq \Omega(\mathcal{C}_*)$$

*where $\Omega(\mathcal{C})$ is the image of the loopspace functor of $\mathcal{C}$. Recall $\mathbb{E}_n \simeq \mathbb{E}_1^{\otimes n}$ (this is the tensor product of operads), in particular we have*

$$\mathbb{E}_n\mathrm{Grp}(\mathcal{C}) \simeq \Omega^n(\mathcal{C}_*)$$

*Formally this applies to the case $n = \infty$ where $\Omega^\infty$ is the infinite loopspace functor $\Omega^\infty : \mathbf{Sp}(\mathcal{C}) \to \mathcal{C}$.*
*Moreover, these equivalences are determined by the adjunction*

$$B^n : \mathbb{E}_n\mathrm{Mon}(\mathcal{C}) \rightleftarrows \mathcal{C}_* : \Omega^n$$

*where $B$ is the bar construction. Again, this applies formally to the case where $n = \infty$, where the adjunction is upgraded to*

$$B^\infty : \mathbb{E}_\infty\mathrm{Mon}(\mathcal{C}) \rightleftarrows \mathbf{Sp}(\mathcal{C}) : \Omega^\infty$$

*Proof.* Omitted. □

**Construction 16.1.3.** By theorem 16.1.2, we have the unit of the adjunction $B^n \dashv \Omega^n$ the left adjoint to the forgetful functor $\mathbb{E}_n\mathrm{Grp}(\mathcal{C}) \to \mathbb{E}_n\mathrm{Mon}(\mathcal{C})$, which we will call the group completion functor, denoted by $X \mapsto X^{\mathrm{grp}}$. Moreover, we have a natural tranfromation $\mathrm{id} \to \Omega^n B^n$, hence we have a natural map $X \to X^{\mathrm{grp}}$.

**Definition 16.1.4.** For any $\mathbb{E}_\infty$ ring $R$ in any stable presentable category $\mathcal{C}$, we consider the category $\mathbf{Mod}_R$. There is a tensor product on this category given by the bar construction, so $\mathbf{Mod}_R$ is a commutative monoid in the category $\mathbf{Cat}^\times$ (the category of $(\infty, 1)$-categories equipped with the Cartesian symmetric monoidal structure). Thus since $\mathrm{Core} : \mathbf{Cat}^\times \to \mathbf{An}^\times$ is symmetric monoidal, we consider the commutative monoid $\mathrm{Core}\,\mathbf{Mod}_R \in \mathbf{An}^\times$. Finally we define the $K$-theory space to be the commutative monoid $K(R) := \mathrm{Core}\,\mathbf{Mod}_R^{\mathrm{grp}}$. We often view this as a connective spectrum due to the isomorphism in theorem 16.1.2.

## 16.2 Waldhausen S-construction

# Chapter 17

# Topological Hochschild homology

## 17.1 Common definitions and first properties

**Theorem 17.1.1.** *For an $\mathbb{E}_n$-ring $R$ in any symmetric monoidal category $\mathcal{C}$, we have* $\mathrm{THH}(R)$ *an $\mathbb{E}_{n-1}$ ring.*

## 17.2 Generalized Thom spectra and the proof of Bokstedt theorem

Recall that in the above we defined the algebraic $K$-theory of a $\mathbb{E}_\infty$ ring $R$ to be $\mathrm{Core}\,\mathbf{Mod}_R^{\mathrm{grp}}$. Here we apply the right adjoint of the forgetful functor $\mathrm{Grp} \to \mathrm{Mon}$ to make the monoid $\mathrm{Core}\,\mathbf{Mod}_R$ a group, which we will denote by the Picard space $\mathrm{Pic}(R)$.

**Definition 17.2.1.** We define the Picard space of an $\mathbb{E}_n$-ring $R$ ($n \geq 2$) in some symmetric monoidal category $\mathcal{C}$ written out by $\mathrm{Pic}(R) := \mathrm{Core}\,\mathbf{Mod}_R^{\mathrm{inv}}$ to be the subspace of $\mathrm{Core}\,\mathbf{Mod}_R$ consisting of tensor-invertible modules (that is, modules $M$ such that there is some $N$ such that $M \otimes_R N \simeq R$) in $\mathrm{Core}\,\mathbf{Mod}_R$. Notice that $\mathbf{Mod}_R$ is $\mathbb{E}_{n-1}$-monoidal, hence $\mathrm{Pic}(R)$ is an $\mathbb{E}_{n-1}$-ring in $\mathbf{An}$. When $R$ is $\mathbb{E}_\infty$, this space is a $\mathbb{E}_\infty$-group, and we sometimes call this the Picard spectrum.
More generally, for any $\mathbb{E}_n$-monoidal category $\mathcal{C}$, we define $\mathrm{Pic}(\mathcal{C}) := \mathrm{Core}\,\mathcal{C}^{\mathrm{inv}}$ an $\mathbb{E}_n$-group in $\mathbf{An}$.

*Remark* 24. For the case where $R$ is an $\mathbb{E}_\infty$-ring spectrum, $\mathrm{Pic}(R)$ coincides with the spectrum $\mathrm{bgl}_1(R)$, which is the suspension of the unit spectrum of $R$. This is by definition, since $\Omega\,\mathrm{Pic}(R)$ is exactly the automorphisms of $R$ viewed as an $R$-module.

**Definition 17.2.2.** Let $X$ be a space and let $R$ be an $\mathbb{E}_n$-ring in some stable presentable category $\mathcal{C}$. Then let $f : X \to \mathrm{Pic}(R)$ be a map of spaces. We define the Thom object of $f$ to be

$$Mf := \varinjlim(X \xrightarrow{f} \mathrm{Pic}(R) \hookrightarrow \mathbf{Mod}_R)$$

**Example 7.** *We let $F(n)$ denote the auto-homotopy equivalences of $S^n$, and let $F = \varinjlim F(n)$. Thus $F$ can be viewed as $GL_1(S)$, where $S$ is the sphere spectrum. Hence $BF \simeq \mathrm{Pic}(S)$ by the above remark.*
*For the canonical bundles on the classifying spaces $BO$, $BU$, and $BSp$, we have the Thom spectra $MO$, $MU$, and $MSp$. They can be obtained from the above construction by the following process. Notice that the $J$-homomorphism $J : O \to F$ is a morphism of groups, hence admit a delooping $BJ : BO \to BF \simeq \mathrm{Pic}(S)$. Then we have $MO \simeq MBJ$. The Thom spectra for $BU$ and $BSp$ is obtained similarly. This gives us a way to think of bundles as maps to $\mathrm{Pic}(S)$.*

**Theorem 17.2.3.** *For $R$ an $\mathbb{E}_n$-ring, the functor $M : \mathbf{An}_{/\operatorname{Pic}(R)} \to \mathbf{Mod}_R$ is $\mathbb{E}_{n-1}$-monoidal. Here $\mathbf{An}_{/\operatorname{Pic}(R)}$ is equipped with the symmetric monoidal structure given by for $f : X \to \operatorname{Pic}(R)$ and $g : Y \to \operatorname{Pic}(R)$, $f \times g : X \times Y \to \operatorname{Pic}(R) \times \operatorname{Pic}(R) \to \operatorname{Pic}(R)$.*

*Proof.* We first notice that by construction Pic is a functor

$$\operatorname{Pic} : \mathbb{E}_{n-1}\mathrm{Mon}(\mathbf{Pr}^{\mathrm{L}}) \to \mathbb{E}_{n-1}\mathrm{Grp}(\mathbf{An})$$

Then we construct a left adjoint to this functor. We claim that there is an adjunction

$$\mathbf{An}_{/-} : \mathbb{E}_{n-1}\mathrm{Grp}(\mathbf{An}) \rightleftarrows \mathbb{E}_{n-1}\mathrm{Mon}(\mathbf{Pr}^{\mathrm{L}}) : \operatorname{Pic}$$

To verify this, notice that $\mathbf{An}_{/G} \simeq \operatorname{Fun}(G, \mathbf{An})$ by the straightening and unstraightening equivalence, hence $\operatorname{Fun}^L(\mathbf{An}_{/G}, \mathcal{C}) \simeq \operatorname{Fun}(G, \mathcal{C})$, and a functor on the left is $\mathbb{E}_{n-1}$-monoidal iff the functor on the right is $\mathbb{E}_{n-1}$-monoidal, hence since $G$ is an $\mathbb{E}_{n-1}$-group, any $\mathbb{E}_{n-1}$-monoidal functor $G \to \mathcal{C}$ factors through $G \to \operatorname{Pic}(\mathcal{C})$. To summarize, we have the chain of equivalences which exhibits the adjunction.

$$\operatorname{Fun}^L_{\mathbb{E}_{n-1}}(\mathbf{An}_{/G}, \mathcal{C}) \simeq \operatorname{Fun}_{\mathbb{E}_{n-1}}(G, \mathcal{C}) \simeq \operatorname{Fun}_{\mathbb{E}_{n-1}}(G, \operatorname{Pic}(\mathcal{C}))$$

With this adjunction, it suffices to see that $M$ is the dual of the identity on $\operatorname{Pic}(R)$. But by construction, the dual of the identity on $\operatorname{Pic}(R)$ is given by the following process due to the above chain of equivalence

$$(\mathbf{An}_{/\operatorname{Pic}(R)} \xrightarrow{\varinjlim} \mathbf{Mod}_R) \leftarrow\!\shortmid (\operatorname{Pic}(R) \hookrightarrow \mathbf{Mod}_R) \leftarrow\!\shortmid \mathrm{id}_{\operatorname{Pic}(R)}$$

which is exactly the construction of Thom object. □

**Corollary 17.2.4.** *For $f : G \to \operatorname{Pic}(R)$ where $G$ is an $\mathbb{E}_\infty$-monoid and $f$ is an $\mathbb{E}_\infty$-map, we have $Mf$ an $\mathbb{E}_\infty$-$R$-algebra.*

**Theorem 17.2.5.** *Let $X$ be a pointed space and $G$ an $\mathbb{E}_\infty$-group of spaces. Then for an $\mathbb{E}_\infty$ ring $R$ and a $\mathbb{E}_\infty$-map $f : G \to \operatorname{Pic}(R)$, we have an equivalence of $\mathbb{E}_\infty$-$R$-algebras*

$$X \otimes Mf \simeq Mf \otimes S[X \otimes_* G]$$

*Here the first tensor denote the tensoring with spaces in the category of $\mathbb{E}_\infty$-$R$-algebras, the second tensor is the coproduct in the category of $\mathbb{E}_\infty$-$R$-algebras, and the third tensor is the tensoring with spaces in the category of $\mathbb{E}_\infty$-groups of spaces. $S[X \otimes G]$*

*Proof.* We first prove that we have the following equivalence

$$X \otimes Mf \simeq M(X \otimes G \to G \to \operatorname{Pic} R)$$

heret □

We now consider the interaction of topological Hochschild homology and Thom spectra, which will present a proof (different from the original one) of the Bokstedt theorem.

# Part V

# Unstable Chromatic Homotopy Theory

# Chapter 18

# Chromatic localizations on spaces

## 18.1 More on chromatic localizations

We first give a short review on reflective localizations.

**Definition 18.1.1.** For a family of morphisms $S$ in a category $\mathcal{C}$, we define an object $X$ to be $S$-local if for any $f : Y \to Z$ in $S$, we have $f^* : \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Y, X)$ is an equivalence. We define $f : Y \to Z$ to be an $S$-equivalence if for all $X$ to be $S$-local, we have $f^* : \mathrm{Hom}(Y, X) \to \mathrm{Hom}(Z, X)$ an equivalence. Finally we define an object $X$ to be $S$-acyclic if $X \to *$ is an $S$-equivalence.

**Theorem 18.1.2.** *There is a localization functor $L_S : \mathcal{C} \rightleftarrows \mathcal{C}_S : \iota$ from $\mathcal{C}$ to the full-subcategory consisting $S$-local objects $\mathcal{C}_S$. Moreover, this localization functor exhibits an equivalence $\mathcal{C}_S \simeq \mathcal{C}[S^{-1}]$. By construction, we have a morphism $X \to L_S(X)$ which is an $S$-equivalence.*

**Definition 18.1.3.** For any object $A$, we define the localization functor $P_A$ to be the reflective localization associated to the set consisting of a single map $A \to *$.

We say an object $X$ is $A$-null if we have $\mathrm{Map}(*, X) \to \mathrm{Map}(A, X)$ is an equivalence. This, equivalently, means $X$ is $S$-local where $S$ consists of the single morphism $A \to *$.

We say an object $Y$ is $A$-periodic if for all $A$-null spectra $X$, the inclusion $\mathrm{Map}(*, X) \to \mathrm{Map}(Y, X)$ is an equivalence. This means the morphism $Y$ is $S$-acyclic where $S$ is the same as in the above.

In particular, the above definitions apply to the category **Sp** or **An**.

*Remark* 25. The theory of Bousfield localizations is complete contained in reflective localizations. Note that for a spectra, $E$-equivalences ($E$-local objects) are simply the $S$-equivalences ($S$-local objects) in which $S$ is the set of morphisms $X \to *$ where $X$ runs through all $E$-acyclic spectra.

**Lemma 18.1.4.** *For any spectrum $A$, $P_A$ commutes with finite colimits, hence for finite $F$, we have $P_A(X) \wedge F \simeq P_A(X \wedge F)$.*

*Proof.* Similar to the proof of theorem 6.1.16. □

**Lemma 18.1.5.** *For $E \wedge A \simeq *$, we have a natural equivalence $E \wedge X \to E \wedge P_A(X)$ for all $X$.*

*Proof.* From the construction of $P_A$ (which I won't do here). (Yes, absolutely not). □

**Definition 18.1.6.** We define $F(n)$ to be a spectrum of type $n$. This is well-defined up to Bousfield equivalence.

**Lemma 18.1.7.** *For a spectrum $X$ and a self map $f : \Sigma^d X \to X$, we have $\langle X \rangle = \langle X[f^{-1}] \rangle \wedge \langle X/\Sigma^d X \rangle$.*

*Proof.* For any spectrum $Y$, $Y \wedge X \simeq *$ obviously implies $Y \wedge X[f^{-1}] \simeq *$ and $Y \wedge X/\Sigma^d X \simeq *$. For the converse, if $Y \wedge X/\Sigma^d X = 0$, then we have a natural equivalence $Y \wedge f : Y \wedge \Sigma^d X \to Y \wedge X$. Thus every map in the diagram $Y \wedge X \to Y \wedge \Sigma^{-d} X \to \cdots$ is an equivalence, hence induces an equivalence $Y \wedge X \to Y \wedge X[f^{-1}]$. Thus $Y \wedge X \simeq *$. $\square$

**Lemma 18.1.8.** *We have an orthogonal decomposition for any choice of $F(n)$*

$$\langle S \rangle = \langle F(0)[v_0^{-1}] \rangle \vee \cdots \vee \langle F(n)[v_n^{-1}] \rangle \vee \langle F(n+1) \rangle$$

*Proof.* By lemma 18.1.7, we have the decomposition. Then, to prove that it is orthogonal, it suffices to prove for $i < j$, $F(i)[v_i^{-1}] \wedge F(j)$ is contractible, so that $F(i)[v_i^{-1}] \wedge F(j)[v_j^{-1}] \simeq *$. However, supposing $v_i$ has degree $d$, this follows from that $v_i \wedge \mathrm{id} : \Sigma^d F(i) \wedge F(j) \to F(i) \wedge F(j)$ is zero on all $K(m)$, hence is nilpotent. $\square$

**Definition 18.1.9.** We define $T(n)$ to be a spectrum of the form $X[f^{-1}]$, where $X$ is of type $n$ and $f$ is a $v_n$-self map of $X$.

**Proposition 18.1.10.** *Such $T(n)$ is well-defined up to Bousfield equivalence.*

*Proof.* For a type $n$ spectrum $X$ with a $v_n$-self map $f$, consider the full subcategory $\mathcal{D}_X$ of finite $p$-local spectra spanned by spectra $Y$ admitting a $v_n$-self map $g$ satisfying $X[f^{-1}] \wedge Z \simeq * \implies Y[g^{-1}] \wedge Z \simeq *$.

We then prove that $\mathcal{D}_X$ is a thick subcategory of finite $p$-local spectra. Now, $\mathcal{D}_X$ is obviously closed under retracts, since for $Y_1$ and $Y_2$ with morphisms $Y_1 \to Y_2 \to Y_1$, we may assume the $v_n$-self maps of $Y_i$ agrees with the morphisms. Then it follows that $Y_1[g_1^{-1}] \wedge Z$ is a retract of $Y_2[g_2^{-1}] \wedge Z$, hence is trivial. Then, for a fiber sequence $Y_1 \to Y_2 \to Y_3 \to Y_4$ with $Y_2$ and $Y_3$ in $\mathcal{D}_X$, we may again assume the $v_n$-self maps of $Y_i$ agrees with this fiber sequence. Thus we have $Y_1[g_1^{-1}] \wedge Z$ and $Y_4[g_4^{-1}] \wedge Z$ are fibers or cofibers of amorphism between trivial spectra, hence trivial.

Finally, by theorem 11.4.9 and that $\mathcal{D}_X$ contains $X$, we deduce that $\mathcal{D}_X \simeq \mathcal{C}_{\geq n}$. Therefore $T(n)$ is well-defined. $\square$

**Corollary 18.1.11.** *We have $L_n^f \simeq L_{T(0) \vee T(1) \vee \cdots \vee T(n)}$.*

*Proof.* Recall that by theorem 11.5.6, we have $L_n^f$ is the smashing localization having the minimal kernel with $\mathbf{Sp}_{(p)}^{\mathrm{fin}} \cap \ker L_{E(n)} = \mathcal{C}_{\geq n+1}$. Then it suffices to prove $L_{T(0) \vee \cdots \vee T(n)}$ has the same property. Firstly, notice that by lemma 18.1.8, we have $\langle S \rangle = \langle T(0) \rangle \vee \cdots \langle T(n) \rangle \vee \langle F(n+1) \rangle$. Assuming for a spectrum $E$, we have $\mathbf{Sp}_{(p)}^{\mathrm{fin}} \cap \ker L_{E(n)} \simeq \mathcal{C}_{\geq n+1}$, then smashing $E$ with the above equality, we have

$$\langle E \rangle = \langle E \wedge T(0) \rangle \vee \cdots \langle E \wedge T(n) \rangle$$

Then, it suffices to show that $L_{T(0) \vee \cdots \vee T(n)}$ is smashing. Notice that we have $\mathbf{Sp}_{(p)}^{\mathrm{fin}} \cap \ker L_E = \mathbf{Sp}_{(p)}^{\mathrm{fin}} \cap \ker L_{L_E(S)}$ for any spectrum $E$. Therefore we may apply $E = L_{T(0) \vee \cdots \vee T(n)}(S)$ to the above equality. In particular, we have

$$\langle L_{T(0) \vee \cdots \vee T(n)}(S) \rangle \leq \langle T(0) \vee \cdots \vee T(n) \rangle$$

However, since each $T(0) \vee \cdots \vee T(n)$-local spectrum is a module over $L_{T(0) \vee \cdots \vee T(n)}(S)$, $T(0) \vee \cdots \vee T(n)$-local implies $L_{T(0) \vee \cdots \vee T(n)}(S)$-local. Therefore, they have the same Bousfield class and hence $L_{T(0) \vee \cdots \vee T(n)}$ is smashing. $\square$

**Theorem 18.1.12.** *We have $L_n^f \simeq P_{F(n+1)}$.*

*Proof.* By lemma 18.1.8, $(T(0)\vee\cdots\vee T(n))\wedge F(n+1)\simeq *$, thus $F(n+1)$ is $T(0)\vee\cdots\vee T(n)$-acyclic. Thus for all $X$, $\mathrm{Map}(F(n+1),L_n^f(X))\simeq *$. By the universal property of $P_{F(n+1)}$, this gives a natural transformation $\mathrm{id}\to P_{F(n+1)}\to L_n^f$. Thus to show the natural transformation is an equivalence, it suffices to show that for all $X$, $P_{F(n+1)}(X)$ is $T(0)\vee\cdots\vee T(n)$-local. To show this, let $Z$ be a $T(0)\vee\cdots\vee T(n)$-acyclic spectrum. We show that any map $Z\to P_{F(n+1)}(X)$ is null. It suffices to show that $P_{F(n+1)}(Z)$ is contractible.

By lemma 18.1.8, we have $T(i)\wedge F(n+1)\simeq *$ for $i\le n$ and all possible choices of $T$ and $F$. Thus by lemma 18.1.5, we have $T(i)\wedge Z\to T(i)\wedge P_{F(n+1)}(Z)$ an equivalence. Since $Z$ is $T(0)\vee\cdots\vee T(n)$-acyclic, it is $T(i)$-acyclic. Therefore $T(i)\wedge P_{F(n+1)}(Z)\simeq *$. Then by definition of nullification, we have $\mathbb{D}\,F(n+1)\wedge P_{F(n+1)}(Z)\simeq \mathrm{Map}(F(n+1),P_{F(n+1)}(Z))\simeq *$. But since $\langle \mathbb{D}\,F(n+1)\rangle\simeq\langle F(n+1)\rangle$ and the equality in lemma 18.1.8, we actually see that $P_{F(n+1)}(Z)$ is $S$-null, hence $P_{F(n+1)}(Z)\simeq *$. □

## 18.2 Localizations in the category $\mathbf{An}_*$

In this section, we apply the localizations introduced in the previous section to the category $\mathbf{An}_*$. The localizations in $\mathbf{An}_*$ is mostly the same as in $\mathbf{Sp}$. A first remark here is that we would very much like to consider the localizations in the category $\mathbf{An}$ instead of $\mathbf{An}_*$ since it would be much more convenient to define the localizations (see the first remark below). However, there is no space $X$ with $\Sigma^\infty_+X$ a type-$n$ spectrum since it must contain a direct summand of the sphere spectrum $S$. On the otherhand, there is clearly a pointed space $Y$ such that $\Sigma^\infty Y$ is of type $n$ since we may consider the cellular structure of $F(n)$, and since it is finite, it must come from a finite pointed CW complex. However, when we pass from the stable case to the unstable case, another problem we may encounter is that spectra representing the same Bousfield class (i.e. different spectra of type $n$), may end up having different Bousfield classes in $\mathbf{An}_*$ (if we regard $F(n)$ as a space with its suspension spectrum type $n$). This would be partially resolved by Bousfield's theorem (theorem 18.3.5), which mainly states that the unstable Bousfield classes of finite type $n$-spaces admitting certain properties is determined by the connectivity of the space.

**Construction 18.2.1.** In the category $\mathbf{An}_*$, the notion of $S$-local spaces and $S$-equivalences must be slightly modified. For pointed spaces $A$ and $B$, we let $\mathrm{Hom}(A,B)$ denote the mapping space of unpointed maps and $\mathrm{Hom}_*(A,B)$ denote the mapping space of pointed maps. Then we define the $S$-local spaces and $S$-equivalences in $\mathbf{An}_*$ to be respectively the objects $X$ such that $\mathrm{Hom}(B,X)\to\mathrm{Hom}(A,X)$ is an equivalence for all $A\to B\in S$ and the morphisms $Y\to Z$ such that $\mathrm{Hom}(Z,X)\to\mathrm{Hom}(Y,X)$ is an equivalence for all $S$-local space $X$.

*Remark* 26. For a self map $f:\Sigma^dA\to A$, the local objects associated to $f$ can be viewed as $Y$ such that $f_*:[A,Y]\to[\Sigma^dA,Y]$. This, in some sense, makes $L_f$ somewhat of a periodization.

We first present a few direct examples on localizations.

**Example 8.** *Let $A=S^n$. Then a space is $A$-null iff it is $(n-1)$-truncated, and is $A$-periodic iff it is $n$-connective.*

*let $A=S^1/p$, then a simply connected $p$-local space is $A$-null iff it is rational, and is $A$-periodic iff its homotopy groups are torsion.*

We might first notice that in the definition above, the choice of the Hom not preserving the base points is not that essential since we may always choose a base point.

**Lemma 18.2.2.** *Let $S$ be a set of morphisms in $\mathbf{An}_*$ consisting of morphisms of connected spaces. Then a space $X$ is $S$-local iff for every choice of base point in $X$, the map $\mathrm{Hom}_*(B,X)\to\mathrm{Hom}_*(A,X)$ is an equivalence for all $A\to B$ in $S$.*

**Lemma 18.2.3.** *Let $S$ be a set of morphisms in $\mathbf{An}_*$. For a morphism of fibrations $f:(p_1: X_1 \to B) \to (p_2 : X_2 \to B)$ inducing an $S$-equivalence (for some fixed set of morphisms $S$) for each $f_* : p_1^{-1}(b) \to p_2^{-1}(b)$, we have $f : X_1 \to X_2$ is an $S$-equivalence.*

*Proof.* Notice that we have natural identifications $X_i \simeq \varinjlim_{b\in B} p_i^{-1}(b)$, then the theorem follows by the fact that $S$-equivalences are preserved under colimits. □

**Lemma 18.2.4.** *Suppose we have a H-group homomorphism $G_1 \to G_2$ and a map of principal bundles.*

$$\begin{tikzcd} G_1 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & B_1 \arrow[d] \\ G_2 \arrow[r] & X_2 \arrow[r] & B_2 \end{tikzcd}$$

*Moreover, the map $X_1 \to X_2$ is required to be $G_1$-equivariant. If $G_1 \to G_2$ and $X_1 \to X_2$ are $S$-equivalences, then we have $B_1 \to B_2$ an $S$-equivalence.*

*Proof.* Simply write $B_i$ as the geometric realization of the simplicial object $X_1 \times (G_1)^n$. □

**Corollary 18.2.5.** *If $X \to Y$ is a map of $n$-connective spaces having the property that $\Omega^n X \to \Omega^n Y$ is an $S$-equivalence, then $X \to Y$ is an $S$-equivalence.*

*Proof.* Directly consider the path fibrations of $\Omega^{k-1}X$ and $\Omega^{k-1}Y$ and the corollary follows by induction. □

**Lemma 18.2.6.** *If $Y$ is $S$-local, then $Y$ is $X \wedge S$-local for all spaces $X$, where $X \wedge S$ is the set of morphisms consisting of $X$ smashed with the morphisms in $S$.*

*Proof.* Let $f : A \to B$ be a morphism in $S$. Then the space $X \wedge A$ is the homotopy cofiber of the map $X \vee A \hookrightarrow X \times A$. This induces the space $\mathrm{Map}(X \wedge A, Y)$ as the homotopy pullback

$$\begin{tikzcd} \mathrm{Hom}(X\wedge A, Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Hom}(X \times A, Y) \arrow[d] \\ Y \arrow[r] & \mathrm{Hom}(X \vee A, Y) \end{tikzcd}$$

Moreover $\mathrm{Hom}(X \vee A, Y)$ fits in a homotopy pullback square

$$\begin{tikzcd} \mathrm{Hom}(X\vee A, Y) \arrow[r] \arrow[d] & \mathrm{Hom}(X, Y) \times \mathrm{Hom}(A, Y) \arrow[d] \\ Y \arrow[r] & Y \times Y \end{tikzcd}$$

Thus it suffices to show that $\mathrm{Hom}(X \times B, Y) \to \mathrm{Hom}(X \times A, Y)$ is a weak equivalence. Rewriting this in terms of $\mathrm{Hom}(X, \mathrm{Hom}(B, Y)) \to \mathrm{Hom}(X, \mathrm{Hom}(A, Y))$ gives the proof immediately. □

The lemmas above gives rise to many corollaries on localizations, which we will list below.

**Theorem 18.2.7.** *In the following list we assume $S$ and $T$ to be sets of morphisms in $\mathbf{An}_*$ and $X$ a pointed space.*

1. *If $S$ consists of morphisms of connected spaces, then $X$ is $S$-local iff every path-component of $X$ is $S$-local (obvious).*

2. *Suppose $S$ consists of morphism of connected spaces, then $L_S X$ is connected implies that $X$ is connected (follows from the construction).*

3. *$L_S$ preserves finite products (follows from lemma 18.2.6 and some theory on exponential ideals).*

4. *$L_S$ preserves colimits (follows from the fact that $L_S$ is a left adjoint).*

5. *A space is $S$-local and $T$-local iff it is $S \vee T$-local, where $S \vee T$ is the set of morphisms of the form $f \vee g$ where $f \in S$, $g \in T$ (direct).*

6. *A space $X$ is $FS$-local iff for every choice of basepoint $GX$ is $S$-local, where $F : \mathbf{An}_* \rightleftarrows \mathbf{An}_* : G$ is a pair of adjoint functors (follows from lemma 18.2.2).*

7. *If $X$ is $S$-acyclic, then $\Sigma X$ is $\Sigma S$-acyclic (follows from the properties of left adjoints).*

8. *Let $f : X \to Y$ be a map of spaces. Assume $Y$ is $S$-local, then $X$ is $S$-local implies that for every point $y$ of $Y$ the fiber $f^{-1}(y)$ is $S$-local (by the long exact sequence of homotopy groups of fiber sequences).*

9. *Let $f : X \to Y$ be a map of spaces. Assume $Y$ is $A$-null, then $X$ is $A$-null iff for every point $y$ of $Y$ the fiber $f^{-1}(y)$ is $A$-null (similar to the above argument).*

10. *If $A$ is $n+1$-connective and $X$ is $n$-truncated, then $X$ is $A$-null (by the properties of truncatedness and connectivity).*

11. *If $S$ consists of $n$-connective maps, then $X \to L_S X$ is $n$-connective (by the construction of $L_S X$).*

Then we introduce some less trivial facts on localizations.

**Lemma 18.2.8.** *Let $S$ be a set of morphisms in $\mathbf{An}_*$, and $W$ is $S$-acyclic. Then for any space $X$, $X$ is $W$-periodic implies that $X$ is $S$-acyclic. In particular, if $W$ is $S$ acyclic having $\pi_0(W) \neq *$, then for all $X$ we have $X$ is $S$-acyclic.*

*Proof.* We prove that if $Y$ is $S$-local, then $Y$ is $W$-null. Since $L_S W \simeq *$, by the universal property of the localization the inclusion $Y \hookrightarrow \mathrm{Hom}(W, Y)$ is an equivalence, hence $Y$ is $W$-null. For the second assertion we notice $S^0$ is a retract of $W$, hence $S^0$ is $S$-acyclic. Thus any $S^0$-periodic space (hence every space) is $S$-acyclic. □

**Lemma 18.2.9.** *Let $S$ be a set of morphisms in $\mathbf{An}_*$. If $L_S X \simeq *$, then for any $k$-connective space $Y$, we have $L_{\Sigma^k S}(X \wedge Y) \simeq *$.*

*Proof.* Firstly the case where $Y \simeq S^n$ is true by entries (4) and (7) in theorem 18.2.7. Then we consider the general case by proving $L_{\Sigma^k S}(X \wedge \tau_{\leq n} Y) \simeq *$ by an induction on $n$. The base case $n = k$ is trivial since $\tau_{\leq k} Y$ is a wedge of $S^k$, and hence is obvious. Then the induction step follows from the fact that we have a cofiber sequence

$$X \wedge \tau_{\leq n-1} Y \to X \wedge \tau_{\leq n} Y \to X \wedge \bigvee S^n$$

Since the first space and last space is $\Sigma^k S$-acyclic, so is the middle space since $L_S$ preserves cofiber sequences.

Finally by taking a colimit $X \wedge Y \simeq \varinjlim X \wedge \tau_{\leq n} Y$, $X \wedge Y$ is $\Sigma^k S$-acyclic. □

**Corollary 18.2.10.** *If $L_S X \simeq *$, then for any $k$-connective spectrum $Z$, we have*

$$L_{\Sigma^k S} \Omega^\infty (X \wedge Z) \simeq *$$

*Proof.* By lemma 18.2.8 it suffices to consider the case where $X$ is connected, otherwise every space is $f$-acyclic. Then since $L_S X \simeq *$, we have $L_{\Sigma^\infty S}\Sigma^\infty X \simeq *$ by a similar argument to entry (7) in theorem 18.2.7, hence $L_S QX \simeq *$, where $QX = \Omega^\infty\Sigma^\infty X$. Therefore to prove $L_{\Sigma^k f}\Omega^\infty(X \wedge Z) \simeq *$, it suffices to prove $L_{\Sigma^{\infty+k} f} X \wedge Z \simeq *$, which is by a similar argument to lemma 18.2.9 $\square$

**Corollary 18.2.11.** *If $\tilde{H}_d(A, \mathbb{F}_p) \neq 0$, then for $d' \geq d$, we have $K(\mathbb{Z}/p\mathbb{Z}, d')$ is $A$-periodic.*

*Proof.* Firstly by letting $X = A$, $Z = H\mathbb{F}_p$, $k = 0$ in corollary 18.2.10, since $P_A A \simeq *$, we have $P_A\Omega^\infty(H\mathbb{F}_p \otimes A) \simeq *$. Then since $\tilde{H}_d(A, \mathbb{F}_p) = G \neq 0$, $\Omega^\infty(H\mathbb{F}_p \otimes A)$ contains $K(G, d)$ as a component in the product. Hence $K(G, d)$ is a retract of $\Omega^\infty(H\mathbb{F}_p \otimes A)$, and hence $P_A K(G, d) \simeq *$. Moreover, we have $G$ an $\mathbb{F}_p$-vector space, so $P_A K(\mathbb{F}_p, d) \simeq *$. Finally by corollary 18.2.5 we have immediately $P_A K(\mathbb{F}_p, d') \simeq *$, since $\Omega^{d'-d} K(\mathbb{F}_p, d') \simeq K(\mathbb{F}_p, d)$ and $\Omega^{d'-d} * \simeq *$. $\square$

**Lemma 18.2.12.** *Let $A$ be a space and $X$ be any space. Then consider the map $X \to P_A X$, and let $F$ be the fiber of the map at any point $p \in P_A X$. Then we have $P_A F \simeq *$.*

*Proof.* By Lurie's straightening and unstraightening construction, the fibration $f : X \to P_A X$ is equivalently a functor $\mathrm{St}_f : P_A X \to \mathbf{An}$, where $f^{-1}(p) \simeq \mathrm{St}_f(p)$. Then we consider the composition $P_A \circ \mathrm{St}_f : X \to \mathbf{An} \to \mathbf{An}$, then this is equivalent to another map to $P_A X$, say $\overline{X} \to P_A X$. Note that the fiber of this map at $p$ is $P_A F$, and we have a morphism of fibrations

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & P_A X \arrow[d] \arrow[dl, dashed] \\
P_A F \arrow[r] & \overline{X} \arrow[r] & P_A X
\end{tikzcd}$$

And since the bottom fibration is of base and every fiber a $W$-null space, hence so is $\overline{X}$ by entry (9) of theorem 18.2.7. Therefore by the universal property of nullification, we get the lift in the diagram. Thus the map $F \to P_A F \to \overline{X}$ is null, which, again by the universal property of nullification, implies that the map $P_A F \to \overline{X}$ is null-homotopic. Moreover, the lift provides a section of the bottom fibration, hence the bottom fibration is trivial, and hence $P_A F$ is itself contractible. $\square$

**Lemma 18.2.13.** *Let $S$ be a set of morphisms in $\mathbf{An}_*$. If $X$ is a product of Eilenberg-Maclane spaces $K(G, n)$ with all $G$ Abelian, then $L_S(X)$ is also a product of Eilenberg-Maclane spaces.*

*Proof.* We first recall from the entry (3) of theorem 18.2.7 that $L_S$ preserves finite products. Then we prove that a space is a product of Eilenberg-Maclane spaces iff it is of the form $\Omega^\infty M$ for some $H\mathbb{Z}$-module $M$. Only if is obvious by the delooping construction. If is by the following observation: for a $H\mathbb{Z}$-module $M$, the composition $M \to H\mathbb{Z} \otimes M \to M$ where the first map is the inclusion $\mathbb{S} \otimes M \to H\mathbb{Z} \otimes M$ induced by the unit of $H\mathbb{Z}$, hence $M$ is a retract of $H\mathbb{Z} \otimes M$. Then choose a Moore spectrum $M_n$ such that $\pi_n(M_n \otimes H\mathbb{Z}) \simeq \pi_n(M)$ and a comparison map $M_n \to M$ inducing the isomorphism on $\pi_n$, and the map $H\mathbb{Z} \otimes (\bigoplus M_n) \to H\mathbb{Z} \otimes M \to M$ induces an isomorphism on homotopy groups, and hence a splitting $M \simeq \bigoplus H\mathbb{Z} \otimes M_n$.

Thus it suffices to see that $L_S$ lifts to the stable case, that is, there is some $\Omega^\infty \tilde{L}_S M \simeq L_S \Omega^\infty M$ for all connective $H\mathbb{Z}$-modules $M$. This follows from the fact that $L_S$ preserves products, hence preseserves infinite loopspaces. $\square$

**Lemma 18.2.14.** *Let $S$ be a set of morphisms in $\mathbf{An}_*$. If $X$ is connected, then $L_S \Omega X \simeq \Omega L_{\Sigma S} X$.*

*Proof.* Firstly we have a map $L_S\Omega X \to \Omega L_{\Sigma S}X$ given by the following process. For a connected space $X$, we have the localization map $X \to L_{\Sigma S}X$. Then apply the loopspace functor we have $\Omega X \to \Omega L_{\Sigma S}X$. Then by the entry (6) in theorem 18.2.7, $\Omega L_{\Sigma S}X$ is $S$-local, hence we have a factorization $\Omega X \to L_S\Omega X \to \Omega L_{\Sigma S}X$.

We then construct its inverse. Since $X$ is connected, so is $L_{\Sigma S}X$. Since $f$ preserves products, the map $\Omega X \to L_S\Omega X$ is a map of loopspaces, and we pass to classifying spaces we have a map $X \to BL_S\Omega X$. Then again by entry (6), $BL_S\Omega X$ is $L_{\Sigma S}$-local, and hence we have a factorization $X \to L_{\Sigma S}X \to BL_S\Omega X$. Applying the loopspace functor we have a map $\Omega L_{\Sigma S} \to L_S\Omega X$, which we would show to be the inverse.

We now verify that the maps are indeed inverses to each other. Since we have a map $\Omega X \to \Omega L_{\Sigma S}X \to L_S\Omega X$ whose composition is the localization of $\Omega X$, by the universal property of localizations $L_S\Omega X$ is also the localization of $\Omega L_{\Sigma S}X$, hence the composition $L_S\Omega X \to \Omega L_{\Sigma S}X \to L_S\Omega X$ is the identity.

For the other composite, we deloop the maps $\Omega X \to L_S\Omega X \to \Omega L_{\Sigma S}X$ and obtain that $X \to BL_S\Omega X \to L_{\Sigma S}X$ is the localization map. Hence we also have the $L_{\Sigma S}X \to BL_S\Omega X \to L_{\Sigma S}X$ is the identity, and passing to loopspaces we obtain the result. □

**Lemma 18.2.15.** *Let $S$ be a set of morphisms in* **An** *consisting of morphisms between connected spaces. Let $X$ be a connected $H$-space, then if $L_SX \simeq *$ then we have $L_{\Sigma S}X$ a product of Abelian Eilenberg-Maclane spaces.*

*Proof.* Firstly notice that $L_{\Sigma S}X$ is a connected $H$-space since $L_{\Sigma S}$ preserves finite products and connectivity (by the entry (3) and (11) in theorem 18.2.7). Moreover, since everything is connected here, we may use the space of pointed maps.

We first notice that we have

$$\mathrm{Hom}_*(\Sigma L_{\Sigma S}X, L_{\Sigma S}X) \simeq \mathrm{Hom}_*(L_{\Sigma S}X, \Omega L_{\Sigma S}X) \simeq \mathrm{Hom}_*(L_{\Sigma S}X, L_SX) \simeq *$$

Thus we have $L_{\Sigma S}X$ is $\Sigma L_{\Sigma S}X$-null. Then we prove $L_{\Sigma S}X$ is a retract of $\mathrm{SP}(L_{\Sigma S}X)$ (the infinite symmetric product), which finishes the theorem since by the Dold-Thom theorem, the symmetric product is a product of Abelian Eilenberg-Maclane spaces and a retract of Abelian Eilenberg-Maclane spaces is a product of Abelian Eilenberg-Maclane spaces. To prove this, we prove the inclusion $L_{\Sigma S}X \hookrightarrow \mathrm{SP}(L_{\Sigma S}X)$ induces an isomorphism

$$\mathrm{Hom}_*(\mathrm{SP}(L_{\Sigma S}X), L_{\Sigma S}X) \simeq \mathrm{Hom}_*(L_{\Sigma S}X, L_{\Sigma S}X)$$

In fact, we claim the following more general lemma. For a connected space $A$ and a connected $\Sigma A$-null $H$-space $X$, we have an equivalence $\mathrm{Hom}_*(\mathrm{SP}(A), X) \simeq \mathrm{Hom}_*(A, X)$. We first notice that since $X$ is $\Sigma A$-null, the space $\mathrm{Hom}_*(A, X)$ is discrete. Moreover $\pi_1(X)$ acts trivially on $\mathrm{Hom}_*(A, X)$ since $X$ is a connected $H$-space. We claim that under these conditions

$$\mathrm{Hom}_*(A \times A, X) \simeq \mathrm{Hom}_*(A, X) \times \mathrm{Hom}_*(A, X)$$

This is obvious since we have $\mathrm{Hom}(A \times A, X) \simeq \mathrm{Hom}(A, \mathrm{Hom}(A, X))$, and we can extract $\mathrm{Hom}_*(A, X)$ from the fiber of $\mathrm{Hom}(A, X) \to X$ by evaluating on the basepoint of $A$. Therefore we have a natural $\mathbb{E}_\infty$-group structure on $\mathrm{Hom}_*(A, X)$. Finally noticing that by Dold-Thom theorem, we have $\mathrm{SP}(A) \simeq \Omega^\infty H\mathbb{Z} \otimes A$ since they are both isomorphic to $\prod K(H_n(A), n)$. Thus it suffices to prove that

$$\begin{aligned}
&\mathrm{Hom}_*(\Omega^\infty(H\mathbb{Z} \otimes A), X) \\
\simeq\, &\mathrm{Hom}(H\mathbb{Z}, B^\infty \mathrm{Hom}_*(A, X)) \\
\simeq\, &\mathrm{Hom}_{\mathbb{E}_\infty}(\mathbb{Z}, \mathrm{Hom}_*(A, X)) \\
\simeq\, &\mathrm{Hom}_*(A, X)
\end{aligned}$$

The only non-trivial part is the first equivalence, which can be proven by considering the following inclusion of full subcategories of $\mathbb{E}_\infty$-groups into the category $\mathrm{Fun}(\mathbf{Fin}_*^{\mathrm{op}}, \mathbf{An})$ (the so-called $\Gamma$-spaces). The first equivalence is then a standard fact by abstract nonsense. □

## 18.3 The unstable Bousfield class and Bousfield theorem

As before, we may introduce the notion of Bousfield classes to the category $\mathbf{An}_*$.

**Definition 18.3.1.** We say two pointed spaces $A$ and $B$ are unstably Bousfield equivalent if the $A$-null spaces coincide with the $B$-null spaces. We denote $\langle A\rangle$ by the set of all spaces unstably Bousfield equivalent to $A$. Moreover, we define $\langle A\rangle \le \langle B\rangle$ if $X$ is $B$-null implies that $X$ is $A$-null.

**Corollary 18.3.2.** *We have $\langle A \wedge B\rangle \le \langle A\rangle$.*

**Lemma 18.3.3.** *The following are equivalent.*

1. $\langle A\rangle \le \langle B\rangle$.
2. *Every $A$-periodic space is $B$-periodic.*
3. *$A$ is $B$-periodic.*
4. *The unit $X \to P_B X$ factors as $X \to P_A X \to P_B X$.*
5. $P_B P_A X \simeq P_B X$.

*Proof.* Direct. □

We then prove the following fundamental theorem on the unstable Bousfield class related to telescopic localizations.

**Condition 18.3.4.** We consider the following condition on a space $A$. We suppose $A$ is a finite $p$-local space of the form $\Sigma B$ with $B$ connected. Moreover we require $A$ to have integral homology bounded $p$-power torsion (which along with the condition $A = \Sigma B$ implies that $A$ is simply connected).

**Theorem 18.3.5** (Bousfield)**.** *Let $A$ and $A'$ be finite $p$-local spaces satisfying condition 18.3.4 and with their suspension spectrum having type $> 0$ (that is, the reduced integral homology is torsion), then we have $\langle A\rangle \le \langle A'\rangle$ iff $\operatorname{type} A \ge \operatorname{type} A'$ (we abbreviate the type of $\Sigma^\infty A$ by $\operatorname{type} A$), and $\operatorname{conn} A \ge \operatorname{conn} A'$.*

We will need the following result to prove theorem 18.3.5

**Theorem 18.3.6.** *Let $A$ be a $n$-connective pointed space such that the reduced homology groups $\tilde{H}_*(A,\mathbb{Z})$ are $p$-power torsion and $H_n(A;\mathbb{F}_p) \ne 0$, then we have for any $k \ge 1$, $\operatorname{fib}(P_{\Sigma^{k+1}A}X \to P_{\Sigma^k A}X)$ is equivalent to an Eilenberg-Maclane space $K(G, n+k)$, where $G$ is a $p$-torsion Abelian group.*

*Proof.* By replacing $A$ by $\Sigma^{k-1}A$, it suffices to show that the fiber $F$ of $P_{\Sigma^2 A}X \to P_{\Sigma A}X$ is an Eilenberg-Maclane space of the form $K(G, n+1)$ with $G$ Abelian and $p$-torsion.

By lemma 18.2.12 we have $P_{\Sigma A}F \simeq *$ since the nullification $P_{\Sigma A}F \simeq P_{\Sigma A}P_{\Sigma^2 A}F$. Moreover we have $P_{\Sigma^2 A}F \simeq F$ since the two spaces are already $\Sigma^2 A$-null. Then we claim $P_{\Sigma^2 A}F$ is an $H$-space, so that we can apply lemma 18.2.15. The structure of an $H$-space on $P_{\Sigma^2 A}F$ originates from the fact that $P_{\Sigma^2 A}(F \vee F) \simeq P_{\Sigma^2 A}(F \times F)$. This is because

$$\operatorname{fib}(F \vee F \to F \times F) \simeq S^1 \wedge \Omega F \wedge \Omega F \simeq \Omega F \wedge Z$$

where $Z$ is simply connected, and $\Omega F \wedge Z$ is $\Sigma^2 A$-null since $P_{\Sigma A}F \simeq * \implies P_A \Omega F \simeq *$, and by lemma 18.2.9 we have $P_{\Sigma^2 A}(\Omega F \wedge Z) \simeq *$. Thus the codiagonal $P_{\Sigma^2 A}(F \vee F) \to P_{\Sigma^2 A}F$ can be lifted to a map $P_{\Sigma^2 A}(F \times F) \to P_{\Sigma^2 A}F$, and hence $P_{\Sigma^2 A}F$ is an $H$-space.

Now that $F$ is a product of Abelian Eilenberg-Maclane spaces, we show that there is in fact only one Abelian Eilenberg-Maclane spaces in the product and that the group is $p$-torsion. By the entry (11) of theorem 18.2.7, and that $A$ is $n$-connective, we have $\Sigma A \to *$ is $n+1$-connective, hence so is $F$.

Next we show $\pi_m(F)$ has no $p$-torsion for all $m \geq n+2$. Notice that $K(\pi_m(F), m)$ is a retract of $F$, hence it is $\Sigma^2 A$-null and $\Sigma A$-periodic. Moreover, by corollary 18.2.11, since

$$\tilde{H}_{n+1}(\Sigma A, \mathbb{F}_p) = \tilde{H}_{n+2}(\Sigma^2 A, \mathbb{F}_p) = \tilde{H}_n(A, \mathbb{F}_p) = 0$$

we have $K(\mathbb{F}_p, m)$ is $\Sigma A$-periodic for $m \geq n+1$ and $\Sigma^2 A$-periodic for $m \geq n+2$. Hence the map $K(\pi_m(F), m) \to K(\pi_m(F)/\pi_m(F)[p], m)$ has fiber $K(V, m)$, where $V$ is a $\mathbb{F}_p$-vector space, hence the map is an equivalence after applying $P_{\Sigma^2 A}$ for $m \geq n+2$. Then since $K(\pi_m(F), m)$ is $\Sigma^2 A$-null, the map is actually an equivalence for $m \geq n+2$. Therefore $\pi_m(F)$ has no $p$-torsion for $m \geq n+2$.

Then we show $\pi_m(F)$ satisfies $\pi_m(F) \otimes \mathbb{Z}[p^{-1}] = 0$ for all $m \geq n+2$. Consider the fiber sequence

$$K(\pi_m(F) \otimes \mathbb{F}_p, m-1) \to K(\pi_m(F), m) \xrightarrow{\times p} K(\pi_m(F), m)$$

Thus since $K(\pi_m(F) \otimes \mathbb{F}_p, m-1)$ is $\Sigma A$-periodic, the multiplication by $p$ map

$$K(\pi_m(F), m) \to K(\pi_m(F), m)$$

is an equivalence after applying $P_{\Sigma A}$. Passing to the limit, we have

$$K(\pi_m(F), m) \to K(\pi_m(F) \otimes \mathbb{Z}[p^{-1}], m)$$

is an equivalence after applying $P_{\Sigma A}$. Therefore $K(\pi_m(F) \otimes \mathbb{Z}[p^{-1}], m)$ is $\Sigma A$-periodic since so is $K(\pi_m(F), m)$. However, $K(\pi_m(F) \otimes \mathbb{Z}[p^{-1}], m)$ is also $\Sigma A$-null by the following argument. Since $A$ has reduced homology consisting of $p$-power torsion, it (and hence $\Sigma A$) has trivial reduced $\mathbb{Z}[p^{-1}]$-homology. By the universal coefficient theorem, it has also trivial $\mathbb{Z}[p^{-1}]$-cohomology. Therefore by the representability of cohomology, we have

$$\mathrm{Hom}(\Sigma A, K(\pi_m(F) \otimes \mathbb{Z}[p^{-1}], m)) \simeq K(\pi_m(F) \otimes \mathbb{Z}[p^{-1}], m)$$

Thus $K(\pi_m(F) \otimes \mathbb{Z}[p^{-1}], m)$ is both $\Sigma A$-null and $\Sigma A$-periodic, hence $\pi_m(F) \otimes \mathbb{Z}[p^{-1}]$ is trivial.

Now we have proven that $\pi_m(F) = 0$ for all $m \geq n+2$ since they have no $p$-torsion and trivial after inverting $p$. It remains to show that $\pi_{n+1}(F)$ is $p$-torsion. Since $K(\pi_{n+1}(F), n+1) \to K(\pi_{n+1}(F)/\pi_{n+1}(F)[p], n+1)$ is an equivalence after applying $P_{\Sigma A}$, $K(\pi_{n+1}(F)/\pi_{n+1}(F)[p], n+1)$ is $\Sigma A$-periodic. Similar to the above argument, notice that $\pi_{n+1}(F)/\pi_{n+1}(F)[p]$ is $p$-torsion free, hence the space is $\Sigma A$-null. Hence $\pi_{n+1}(F)/\pi_{n+1}(F)[p] = 0$, and hence $\pi_{n+1}(F)$ is $p$-torsion. □

*Proof.* We now prove theorem 18.3.5. We first state an auxillary lemma. We claim that if $\Sigma^\infty A$ is contained in the thick subcategory generated by $\Sigma^\infty A'$, then there is some $d$ such that $\langle \Sigma^d A \rangle \leq \langle A' \rangle$. We consider the full subcategory $\mathcal{C}$ of finite $p$-local spectra consisting of spectra of the form $\Sigma^{\infty+n} B$, where $n$ is any integer (may be negative) and $B$ is a finite $p$-local space such that $\langle B \rangle \leq \langle A' \rangle$. Then obviously this category contains $\Sigma^\infty A'$, and it suffices to prove that it is thick, so that it must contain $\Sigma^\infty A$, and hence a suspension of $A$ satisfies the desired property.

For a cofiber sequence of spectra $P \to Q \to R$ with two entries in $\mathcal{C}$, we wish to prove the third is also in $\mathcal{C}$. Now by definition it is closed under arbitrary suspension and loopspaces, so WLOG we assume $P$ and $Q$ are in $\mathcal{C}$ and prove $R$ is in $\mathcal{C}$. Again by definition of $\mathcal{C}$, we suppose there is a cofiber sequence of spaces $L \to M \to N$ such that

$$P \to Q \to R \simeq \Sigma^{\infty+k} L \to \Sigma^{\infty+k} M \to \Sigma^{\infty+k} N$$

moreover we may assume $\langle A'\rangle \geq \langle L\rangle$ and $\langle A'\rangle \geq \langle M\rangle$, or equivalently $L$ and $M$ are $A'$-periodic. Then since $P_{A'}$ is a left adjoint, it preserves cofiber sequences, hence $P_{A'}L \to P_{A'}M \to P_{A'}N$ is isomorphic to a cofiber sequence $* \to * \to P_{A'}N$, which forces $P_{A'}N$ to be trivial.

Then for a retract diagram $P \to Q \to P$ with $Q \in \mathcal{C}$, again assume

$$P \to Q \to P \simeq \Sigma^{\infty+k}L \to \Sigma^{\infty+k}M \to \Sigma^{\infty+k}L$$

with $M$ $A'$-periodic. Then $L$ is $A'$-periodic since $A'$-periodicity is closed under retracts. Hence we have proven the auxillary lemma.

We now prove that if $\langle A\rangle \leq \langle A'\rangle$ then $\operatorname{type} A \geq \operatorname{type} A'$ and $\operatorname{conn} A \geq \operatorname{conn} A'$. For connectivity, notice that since $A'$ is $\operatorname{conn} A'$-connective, it is $S^{\operatorname{conn} A'}$-periodic. Hence we have $\langle A'\rangle \leq \langle S^{\operatorname{conn} A'}\rangle$, hence $A$ is also $\operatorname{conn} A'$-connective by transitivity of the partial order. Thus $\operatorname{conn} A \geq \operatorname{conn} A'$. For the type, assume $\widetilde{K(n)}_*(A') = 0$, since $K(n)$ is a field, equivalently we have $\widetilde{K(n)}^*(A') = 0$, hence

$$\pi_m \operatorname{Hom}(A', \Omega^\infty K(n)) \simeq \pi_0 \operatorname{Hom}(A', \Omega^{\infty+m}K(n)) \simeq \widetilde{K(n)}^{-m}(A') \simeq *$$

and thus $\Omega^{\infty+m}K(n)$ are $A'$-null for all $m$, and hence are $A$-null for all $m$ by assumption. Thus $\widetilde{K(n)}_*(A) = 0$, and hence $\operatorname{type} A \geq \operatorname{type} A'$.

Finally it suffices to prove the converse. Since $\operatorname{type} A \geq \operatorname{type} A'$, by the thick subcategory theorem we clearly have $\Sigma^\infty A$ in the thick subcategory generated by $\Sigma^\infty A'$, hence there is some $d$ such that $\langle \Sigma^d A\rangle \leq \langle A'\rangle$. We will prove that $\langle \Sigma^{d-1}A\rangle \leq \langle A'\rangle$, so that by induction we will have $\langle A\rangle \leq \langle A'\rangle$. Hence we may assume that $d = 1$. By the assumption, $P_{A'}(A) \simeq P_{A'}P_{\Sigma A}(A)$. Hence to show $\langle A\rangle \leq \langle A'\rangle$, it suffices to prove $P_{A'}P_{\Sigma A}(A) \simeq *$. But note that obviuously we have $P_A(A) \simeq *$, hence $P_{\Sigma A}(A)$ is the fiber of the map $P_{\Sigma A}(A) \to P_A(A)$, which, by theorem 18.3.6, has the form $K(G, d')$ where $G$ is $p$-power torsion. But this is clearly $A'$-periodic by corollary 18.2.11. Hence we have proven the theorem. □

# Chapter 19

# The Bousfield-Kuhn functor

## 19.1 Unstable telescopic localizations

With the preparations in the previous sections, we can finally begin the study of telescopic localizations on spaces.

Recall that the telescopic localization $L_n^f$ is given by killing all the $v_m$-maps for $m < n$ and inverting the $v_n$-map. In theorem 18.1.12, we find that $L_n^f \simeq P_{F(n+1)}$, which gives two possible methods to define telescopic localization on spaces. We first focus on one. We remark here that from now on, when we mention a type $n$ space, it will be implicitly be $p$-local for some fixed $p$.

**Definition 19.1.1.** We fix integers $n$ and $d$. Suppose $d$ is sufficiently large so that there exists a space $F(n+1,d)$ of type $n+1$ and connectivity $d$ satisfying condition 18.3.4. Then we define the category $L_n^f(\mathbf{An}_*)_{(p)}^{>d}$ to be the full subcategory of $\mathbf{An}_*$ consisting of the spaces that is $(d+1)$-connective, $p$-local and $F(n+1,d)$-null. By definition, this category admits all limits and colimits.

**Corollary 19.1.2.** *We have a localization functor $P_{F(n+1,d)} : (\mathbf{An}_*)_{(p)}^{>d} \to L_n^f(\mathbf{An}_*)_{(p)}^{>d}$.*

*Proof.* The proof is mainly due to the definition of local objects. We only need to deal with the connectivity conditions here. Notice that $F(n+1,d) \to *$ is $d$-connective, so by the entry (11) of theorem 18.2.7, so is $X \to P_A(X)$. Now $X$ and $P_A(X)$ have isomorphic homotopy groups up to degree $d-1$, and we show $\pi_d(P_A(X)) = 0$. We have an exact sequence of homotopy groups

$$\pi_d(X) \to \pi_d(P_A(X)) \to \pi_{d-1}(X, P_A(X))$$

with the first and last term zero, which forces $\pi_d(P_A(X)) = 0$. □

The difference between the categories $L_n^f(\mathbf{An}_*)_{(p)}^{>d}$ when the type $n$ varies is analogous to the stable case, but we do encounter something new when $d$ varies. A good news is that this difference is simple enough.

**Proposition 19.1.3.** *There is a fully faithful embedding $L_n^f(\mathbf{An}_*)_{(p)}^{>d+1} \hookrightarrow L_n^f(\mathbf{An}_*)_{(p)}^{>d}$ which identifies the image as the spaces $X \in L_n^f(\mathbf{An}_*)_{(p)}^{>d}$ such that $(\pi_{d+1}X)_{\mathbb{Q}}$-vanishes.*

*Proof.* We consider the functor as the composition

$$L_n^f(\mathbf{An}_*)_{(p)}^{>d+1} \hookrightarrow (\mathbf{An}_*)_{(p)}^{>d} \xrightarrow{P_{F(n+1,d)}} L_n^f(\mathbf{An}_*)_{(p)}^{>d}$$

which we will abuse notation to denote by $P_{F(n+1,d)}$. Then for $X \in L_n^f(\mathbf{An}_*)_{(p)}^{>d+1}$, the map $X \to P_{F(n+1,d)}X$ has fiber a $F(n+1,d)$-periodic space by the lemma 18.2.12. Now, by example 8, recall that $S^1/p$-periodic $p$-local spaces have torsion homotopy groups. Then since $\Sigma S^1/p$ have type 1 and connectivity 2, we have $\langle \Sigma S^1/p\rangle \geq \langle F(n+1,d)\rangle$. Thus the fiber is also $\Sigma S^1/p$-periodic, hence trivial on rational homotopy groups. Thus $X \to P_{F(n+1,d)}X$ is an isomorphism on rational homotopy groups, hence we have the image of $P_{F(n+1,d)}$ contained in spaces where the $(d+1)$-th rational homotopy groups zero.

Then we consider the functor $\tau_{\geq d+2} : L_n^f(\mathbf{An}_*)_{(p)}^{>d} \to (\mathbf{An}_*)_{(p)}^{>d+1}$. For $X \in L_n^f(\mathbf{An}_*)_{(p)}^{>d}$, we have a fiber sequence $\tau_{\geq d+2}X \to X \to K(\pi_{d+1}(X), d+1)$. Since $X$ is $F(n+1,d)$-null, it is $\Sigma F(n+1,d)$-null by lemma 18.2.6. For $X$ with $(\pi_{n+1}(X))_{\mathbb{Q}}$ non-trivial, $K(\pi_{n+1}(X), d+1)$ is clearly $\Sigma F(n+1,d)$-null by the definition of spaces of type $n+1$. So when $(\pi_{n+1}(X))_{\mathbb{Q}}$ is non-trivial, $\tau_{\geq d+2}$ upgrades to a functor $L_n^f(\mathbf{An}_*)_{(p)}^{>d} \to L_n^f(\mathbf{An}_*)_{(p)}^{>d+1}$.

Finally we show that for $X$ with $(\pi_{n+1}(X))_{\mathbb{Q}}$-trivial, $\tau_{\geq d+2}$ is an inverse to the functor $P_{F(n+1,d)}$. Firstly for a space $Y \in L_n^f(\mathbf{An}_*)_{(p)}^{>d+1}$, we prove $Y \to P_{F(n+1,d)}(Y)$ induces an isomorphism on homotopy groups in degrees $> d+1$. Let $F$ be its fiber. Since $Y$ is $P_{\Sigma F(n+1,d)}$-local, $F$ is equivalently the fiber of $P_{\Sigma F(n+1,d)}(Y) \to P_{F(n+1,d)}(Y)$, hence has the form $K(G,d)$ by theorem 18.3.6. By delooping we have a fiber sequence $Y \to P_{F(n+1,d)}(Y) \to K(G, d+1)$, so $Y$ must be the $(d+2)$-connective cover of $P_{F(n+1,d)}(Y)$.

Then for a space $X \in L_n^f(\mathbf{An}_*)_{(p)}^{>d}$ with $(\pi_{d+1}(X))_{\mathbb{Q}} = 0$, we prove the map $\tau_{\geq n+2}X \to X$ is a $F(n+1,d)$-nullification of $\tau_{\geq n+2}X$. We show that the fiber of this map is $F(n+1,d)$-periodic. Note that the fiber is the Eilenberg-Maclane space $K(\pi_{d+1}(X), d)$, but $\pi_{d+1}(X)$ is $p$-power torsion, and hence is periodic by corollary 18.2.11. □

**Lemma 19.1.4.** *Let $X$ be a $p$-local space. Then $X$ is $F(n+1,d)$-periodic iff the three following conditions are satisfied.*

1. *$X$ is $d$-connective.*
2. *The homotopy group $\pi_d(X)$ is $p$-power torsion.*
3. *The connective cover $\tau_{\geq d+1}X$ is $\Sigma F(n+1,d)$-periodic.*

*Proof.* If the three conditions are satisfied, then we have a fiber sequence $\tau_{\geq d+1}X \to X \to K(\pi_d(X), d)$, where $K(\pi_d(X), d)$ and $\tau_{\geq d+1}X$ are $F(n+1,d)$-periodic by corollary 18.2.11 and definition, hence $X$ is $F(n+1,d)$-periodic.

Conversely, suppose $X$ is $F(n+1,d)$-periodic. By the entry (11) of theorem 18.2.7 $X \to P_{F(n+1,d)}X \simeq *$ is $d$-connective. We have $P_{\Sigma F(n+1,d)}X$ equivalent to the fiber of $P_{\Sigma F(n+1,d)X} \to P_{F(n+1,d)}X \simeq *$, hence $P_{\Sigma F(n+1,d)}X \simeq K(G,d)$ for some $p$-power torsion Abelian group. Then consider the fiber sequence $F \to X \to P_{\Sigma F(n+1,d)}X \simeq K(G,d)$. Now $F$ is $\Sigma F(n+1,d)$-periodic, hence $(d+1)$-connective. This implies that $F$ is the $(d+1)$-connective cover of $X$, hence the second and third condition follows immediately. □

**Proposition 19.1.5.** *The functor $\mathbf{An}_* \to L_n^f(\mathbf{An}_*)_{(p)}^{>d}$ given by $X \mapsto P_{F(n+1,d)}(\tau_{\geq d+1}X_{(p)})$ is left exact (that is, preserves finite limits). We denote this functor by $L_{n,d}^f$.*

*Proof.* Obviously taking connective covers commutes with limits. Localization at $p$ is left exact since it firstly preserves the final object, and preserves pullbacks due to the Mayer-Vietoris sequence of homotopy groups of pullbacks, and the fact that $\mathbb{Z}_{(p)}$ is flat. Hence it now suffices to prove the functor $P_{F(n+1,d)} : (\mathbf{An}_*)_{(p)}^{>d} \to (\mathbf{An}_*)_{(p)}^{>d}$ is left exact. Again, it obviously preserves final objects, so it suffices to check that it preserves pullbacks.

Consider maps of $d$-connected pointed spaces $X_1 \to X_0 \leftarrow X_2$. We show the canonical map

$$\tau_{\geq d+1}(X_1 \times_{X_0} X_2) \to \tau_{\geq d+1}(P_{F(n+1,d)}X_1 \times_{P_{F(n+1,d)}X_0} P_{F(n+1,d)}X_2)$$

has $F(n+1,d)$-periodic fiber. We let $F$ denote this fiber. Obviously $F$ is $d$-connective, and since rationalization is left exact, $\pi_*(F)_{\mathbb{Q}}$ is trivial, hence the homotopy groups of $F$ is $p$-power torsion. Thus by lemma 19.1.4, by induction, it suffices to find some $P_A$-local space $F'$ such that their $d'$-connective cover agree for $d' \gg d$. To do this, we let $Y_i$ be the fiber of the map $X_i \to P_{F(n+1,d)}X_i$, and consider $F' = \tau_{\geq d+1}Y_1 \times_{\tau_{\geq d+1}Y_0} \tau_{\geq d+1}Y_2$. Now they have a common connctive cover $\tau_{\geq d+1}(Y_1 \times_{Y_0} Y_2)$ by considering the long exact sequence of homotopy groups of a fiber sequence or a pullback square, hence by lemma 19.1.4 it suffices to prove $F'$ is $F(n+1,d)$-periodic.

Notice that

$$P_{F(n+1,d)}\Omega\tau_{d+1}Y \simeq \Omega P_{\Sigma F(n+1,d)}\tau_{d+1}Y \simeq *$$

where the first equivalence is lemma 18.2.14 the second equivalence is by lemma 19.1.4. Thus $\Omega\tau_{d+1}Y_0$ is $F(n+1,d)$-periodic. Then we consider the fiber sequence $K \to \tau_{d+1}Y_1 \to \tau_{d+1}Y_0$. This gives a fiber sequence $\Omega\tau_{d+1}Y_0 \to K \to \tau_{d+1}Y_1$. Hence $K$ is $F(n+1,d)$-periodic since both $\Omega\tau_{d+1}Y_0$ and $\tau_{d+1}Y_1$ is. Now consider the fiber sequence $K \to \tau_{d+1}Y_1 \times_{\tau_{d+1}Y_0} \tau_{d+1}Y_2 \to \tau_{d+1}Y_2$, which proves that $F'$ is $F(n+1,d)$-periodic. This finishes the whole proof. □

We now wish to investigate the properties preserved by $L^f_{n,d}$, which also leads to the other route to defining telescopic localizations in spaces. As before, we start from the stable setting.

**Definition 19.1.6.** For a spectrum $X$ and a finite $p$-local spectrum $F(n)$ of type $n$, we define the $v_n$-periodic homotopy groups of $X$ by

$$v_n^{-1}\pi_d(X;F(n)) \simeq \varinjlim(\pi_d \operatorname{Map}(F(n),X) \xrightarrow{v_n} \pi_{d+t}\operatorname{Map}(F(n),X) \to \cdots)$$

where the colimit is given by precomposing with $v_n$. This construction only depends on the choice of $F(n)$ by corollary 11.4.18.

**Proposition 19.1.7.** *A map of spectra $f : X \to Y$ is an $T(n)$-equivalence iff it induces an isomorphim on $v_n$-periodic homotopy groups for any choice of $F(n)$.*

*Proof.* We prove that we have the following identification of $v_n^{-1}\pi_*$ as a homology theory

$$v_n^{-1}\pi_*(X) \simeq DT(n)_*(X)$$

where $DT(n)$ is the dual of the telescope of $F(n)$. Let $v_n$ be a $v_n$-self map of $F(n)$, then $Dv_n$ is a $v_n$-self map of $\mathbb{D}\,F(n)$. Hence we may upgrade the construction of $v_n^{-1}\pi_d(X;F(n))$ to the colimit of spectrum

$$\varinjlim(\operatorname{Map}(F(n),X) \to \operatorname{Map}(\Sigma^t F(n),X) \to \cdots)$$

By definition of $p$-local finite spectra and duals, this is equivalently the colimit

$$\varinjlim(\mathbb{D}\,F(n)\otimes X \leftarrow \Sigma^{-t}\,\mathbb{D}\,F(n)\otimes X \to \cdots)$$

and by the definition of $T(m)$ in definition 11.5.1, this is obviously $DT(m)\otimes X$. Hence we have the identification above. □

We then turn to the unstable setting.

**Definition 19.1.8.** For a space $X$, and a finite space of type $m$ equipped with a $v_n$-self map $v_n : \Sigma^t F(n) \to F(n)$, we define its $v_n$-periodic homotopy groups $v_n^{-1}\pi_d(X;F(n))$ by

$$v_n^{-1}\pi_d(X;F(n)) \simeq \varinjlim(\pi_d \operatorname{Hom}_*(F(n),X) \xrightarrow{v_n} \pi_{d+t}\operatorname{Hom}_*(F(n),X) \to \cdots)$$

where the colimit is given by precomposing with $v_n$. This construction depends on the choice of $F(n)$, but not the choice of the $v_n$-self map (we will see this later).

**Definition 19.1.9.** We define a map $f : X \to Y$ to be $v_n$-periodic equivalence if for all choices of base points and some $F(n)$, we have $v_n^{-1}\pi_*(X;F(n)) \to v_n^{-1}\pi_*(Y;F(n))$ is an equivalence.

**Construction 19.1.10.** We may realize $v_n^{-1}\pi_d(X;F(n))$ as the homotopy groups of a spectra, since by definition it is periodic of period $t$. Explicitly, the spectra is constructed as follows. Let

$$Y=\mathrm{Hom}_*(F(n),X)\to \mathrm{Hom}_*(\Sigma^tF(n),X)\to \mathrm{Hom}_*(\Sigma^{2t}F(n),X)\to\cdots$$

Then clearly $\Omega^tY=Y$. Hence we define the spectrum $\Phi_{v_n}(X)$ by letting its $k$-th space be $\Omega^{Nt-k}Y$, where $N\gg 0$ is any integer such that $Nt>k$.

Not so obviously, the above construction does not depend on the choice of the $v_n$-self map by the following arguments.

**Lemma 19.1.11.** *Let $v_n:\Sigma^tF(n)\to F(n)$ be a $v_n$-self map of a finite space of type $n$. Then $\Phi_{\mathrm{id}_W\wedge v_n}(X)\simeq \mathrm{Map}(W,\Phi_{v_n}(X))$, $\Phi_{v_n^k}(X)\simeq\Phi_{v_n}(X)$. Here $\mathrm{id}_W\wedge v_n$ is the map $\mathrm{id}_W\wedge v_n: W\wedge\Sigma^tF(n)\to W\wedge F(n)$, where $W$ is a pointed space, and $v_n^k:\Sigma^{kt}F(n)\to F(n)$ be the map obtained by $v_n$ composed with itself by $k$ times. We remark here that by the Kunneth formula for the $K(n)$-homology, $W\wedge F(n)$ has always type $\geq n$, and when it has type $>n$, the $v_n$-self map $\mathrm{id}_W\wedge v_n$ is nilpotent, and $\Phi_{\mathrm{id}_W\wedge v_n}(X)\simeq 0$.*

*Proof.* Direct. □

**Corollary 19.1.12.** *For two different $v_n$-self maps $v_n:\Sigma^tF(n)\to F(n)$ and $v_n':\Sigma^{t'}F(n)\to F(n)$, we have $\Phi_{v_n}\simeq\Phi_{v_n'}$.*

*Proof.* Recall from corollary 11.4.18 that we may assume $v_n$ and $v_n'$ are stably equivalent after raising to a suitable power. Thus $\Sigma^kv_n^a\simeq\Sigma^k(v_n')^b$. By lemma 19.1.11 we have $\Phi_{v_n}\simeq\Sigma^k\Phi_{\Sigma^kv_n^a}\simeq\Sigma^k\Phi_{\Sigma^k(v_n')^b}\simeq\Phi_{v_n'}$. □

Therefore by the uniqueness of $\Phi_{v_n}$, we will denote it by $\Phi_{F(n)}$ to show its independence of the choice of the $v_n$-self map. But do notice that the choice of the type $n$ space $F(n)$ still matters in the construction. We also remark here that the construction $\Phi_{F(n)}(X)$ is also somewhat functorial in the choice of $F(n)$. Namely we have the following property.

**Lemma 19.1.13.** *For a map of type $n$ spaces $F(n)\to F'(n)$, we have a functor $\Phi_{F'(n)}(X)\to\Phi_{F(n)}(X)$. Moreover, if we regard $F''(n)=\mathrm{cof}(F(n)\to F'(n))$ as a space equipped with a $v_n$-self map, then we have*

$$\Phi_{F''(n)}=\mathrm{fib}(\Phi_{F'(n)}(X)\to\Phi_{F(n)}(X))$$

*Proof.* Firstly the functorality is obtained by corollary 11.4.18. The second property is by the fact that a cofiber of a map of $v_n$-self maps is a $v_n$-self map, and taking fibers commute with filtered colimits. □

**Theorem 19.1.14.** *Let $f:X\to Y$ be a pointed map of pointed spaces. Then for any type $n$ spaces $F(n)$ and $F'(n)$ equipped with $v_n$-self maps, then $f$ induces an equivalence $\Phi_{F(n)}(X)\simeq\Phi_{F(n)}(Y)$ iff $\Phi_{F'(n)}(X)\simeq\Phi_{F'(n)}(Y)$.*

*Proof.* It suffices to prove that $X\to Y$ induces an isomorphism on $v_n$-periodic homotopy groups with coefficients in $F'(n)$. Recall that by the thick subcategory theorem, $\Sigma^\infty F'(n)$ is in the thick subcategory generated by $\Sigma^\infty F(n)$. Therefore $\Sigma^{\infty+N}F'(n)$ is generated by $\Sigma^\infty F(n)$ under retracts, suspensions and cofibers under finitely many steps for $N\gg 0$. Therefore to prove the above, it suffices to prove the three special cases.

1. $F'(n)$ is a retract of $F(n)$.
2. $F'(n)$ is a suspension of $F(n)$.
3. For another $F''(n)$ such that $\Phi_{F''(n)}(X)\simeq\Phi_{F''(n)}(Y)$ and a map $F''(n)\to F(n)$, $F'(n)=\mathrm{cof}(F''(n)\to F(n))$.

The first case is obvious since if this is the case then $v_n^{-1}\pi_*(X;F'(n))$ is a retract of $v_n^{-1}\pi_*(X;F(n))$, and a retract of an isomorphism is an isomorphism. The second case is by lemma 19.1.11. Finally the third case is by the fact that

$$\mathrm{Hom}_*(F'(n),X) \simeq \mathrm{fib}(\mathrm{Hom}_*(F(n),X) \to \mathrm{Hom}_*(F''(n),X))$$

and that taking fibers commutes with filtered colimits. $\square$

Therefore we may define the $v_n$-periodic homotopy equivalences to the maps send by $\Phi_{F(n)}$ to equivalences for any type $n$ space $F(n)$.

**Theorem 19.1.15.** *For any type $n$ space $F(n)$ equipped with a $v_n$-self map $v_n$, we have $\Phi_{F(n)}(X)$ is $T(n)$-local.*

*Proof.* Recall that we have from theorem 18.1.12 that $L_n^f \simeq P_{F(n+1)}$ and from corollary 18.1.11 that a space is $T(n)$-local iff it is $L_n^f$-local but not $L_{n-1}^f$-local. Therefore it suffices to see that $\Phi_{F(n)}(X)$ is $\Sigma^\infty F'(n+1)$-acyclic but not $\Sigma^\infty F'(n)$-acyclic for some type $n$ space $F'(n)$ and some type $n+1$ space $F'(n+1)$. The first fact follows from lemma 19.1.11 since $F(n) \wedge F'(n+1)$ is a type $(n+1)$-space, hence $\mathrm{Map}(\Sigma^\infty F'(n+1), \Phi_{F(n)}(X)) \simeq \Phi_{F(n)\wedge F'(n+1)}(X) \simeq 0$. For the second fact, it suffices to show that $\Phi_{F(n)\wedge F'(n)}(X) \neq 0$, which follows from theorem 19.1.14 since if the converse holds, then $X \to *$ would be a $v_n$-periodic equivalence, which implies $\Phi_{F(n)}(X) \simeq 0$. $\square$

**Theorem 19.1.16.** *For any pointed space $X$, for $0 < m \leq n$ and any $d$, we have a canonical equivalence $\Phi_{F(m)}(X) \simeq \Phi_{F(m)}(L_{n,d}^f X)$*

*Proof.* Recall that $L_{n,d}^f : \mathbf{An}_* \to L_n^f(\mathbf{An}_*)_{(p)}^{>d}$ is the composition $X \mapsto P_{F(n+1,d)}(\tau_{\geq d+1}X_{(p)})$. Hence it suffices to verify the three constructions do not change $\Phi_{F(m)}$.

Firstly, we prove $\tau_{\geq d+1}X \to X$ induces an equivalence $\Phi_{F(m)}(\tau_{\geq d+1}X) \simeq \Phi_{F(m)}(X)$, which is obvious since we have

$$\Omega^\infty \Phi_{F(m)}(X) \simeq \varinjlim(\mathrm{Hom}_*(F(m),\Omega^{Nt}X) \to \mathrm{Hom}_*(F(m),\Omega^{Nt+t}X) \to \cdots)$$

passes to a sufficiently large loopspace of $X$, which is not affected by the connective cover.

Then we prove that $X \to X_{(p)}$ induces an equivalence $\Phi_{F(m)}(X) \to \Phi_{F(m)}(X_{(p)})$. This is obtained by considering the pullback square of localizations $X \simeq X_{(p)} \times_{X_\mathbb{Q}} X[p^{-1}]$, which results in the pullback square

$$\begin{tikzcd}
\mathrm{Hom}_*(F(m),X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Hom}_*(F(m),X_{(p)}) \arrow[d] \\
\mathrm{Hom}_*(F(m),X[p^{-1}]) \arrow[r] & \mathrm{Hom}_*(F(m),X_\mathbb{Q})
\end{tikzcd}$$

Now by the universal property of localizations, we have

$$\mathrm{Hom}_*(F(m),X[p^{-1}]) \simeq \mathrm{Hom}_*(F(m)[p^{-1}],X[p^{-1}])$$

and hence vanishes since $F(m)$ is of type $> 0$. The likewise fact holds for $\mathrm{Hom}_*(F(m),X_\mathbb{Q})$, hence the bottom arrow is an equivalence, which proves that the top arrow is an equivalence. Thus we have $\Phi_{F(m)}(X) \simeq \Phi_{F(m)}(X_{(p)})$.

Finally it suffices to prove that for any $d$-connected space, the localization $X \to P_{F(n+1,d)}X$ induces an equivalence $\Phi_{F(m)}(X) \to \Phi_{F(m)}(P_{F(n+1,d)}(X))$. Again we may prove only that the $d$-connective cover of the zeroth space of the two spectra are equivalent. Firstly notice that

$L_{T(m)}\Sigma^\infty F(n+1,d) \simeq 0$ since $F(n+1)$ has type $n+1>m$. But by theorem 19.1.15 we see that $\Phi_{F(m)}(X)$ is $T(m)$-local, hence we have

$$\operatorname{Hom}_*(F(n+1,d),\Omega^\infty\Phi_{F(m)}(X)) \simeq \operatorname{Hom}_*(\Sigma^\infty F(n+1,d),\Phi_{F(m)}(X)) \simeq *$$

and hence $\Omega^\infty\Phi_{F(m)}(X)$ is $F(n+1,d)$-null. Therefore the desired equivalence is an exchange map of $P_{F(n+1,d)}$ with the second factor of Hom, hence we can make a few reductions as shown below (we remark here that $\tau_{\geq d+1}$ commutes with filtered colimits since it is a right adjoint).

$$\begin{aligned}
&\tau_{\geq d+1}\Omega^\infty\Phi_{F(m)}(X) \simeq \tau_{\geq d+1}\Omega^\infty\Phi_{F(m)}(P_{F(n+1,d)}X)\\
\iff &\tau_{\geq d+1}P_{F(n+1,d)}\varinjlim(\operatorname{Hom}_*(\Sigma^{kt}F(m),X)) \simeq \Omega^\infty\varinjlim(\operatorname{Hom}_*(\Sigma^{kt}F(m),P_{F(n+1,d)}X))\\
\iff &\varinjlim(\tau_{\geq d+1}(P_{F(n+1,d)}(\operatorname{Hom}_*(\Sigma^{kt}F(m),X)))) \simeq \varinjlim(\tau_{\geq d+1}\operatorname{Hom}_*(\Sigma^{kt}F(m),P_{F(n+1,d)}X))\\
\iff &P_{F(n+1,d)}(\operatorname{Hom}_*(\Sigma^{kt}F(m),X)) \simeq \operatorname{Hom}_*(\Sigma^{kt}F(m),P_{F(n+1,d)}X), \forall k
\end{aligned}$$

But since $F(m)$ is finite, we may write $\Sigma^{kt}F(m)$ as a finite colimit of $S^0$. This makes $\operatorname{Hom}_*(\Sigma^{kt}F(m),X)$ a finite limit, which commutes with $P_{F(n+1,d)}$ by proposition 19.1.5. □

**Proposition 19.1.17.** *Let $X$ be a $F(n+1,d)$-null pointed space. Then for any type $n$ space $F(n)$ equipped with a $v_n$-self map $v_n:\Sigma^tF(n)\to F(n)$, we have the map $\pi_k\operatorname{Hom}_*(F(n),X)\to v_n^{-1}\pi_k(X;F(n))$ an isomorphism for $k>d$.*

*Proof.* We claim that the transition maps in the colimit

$$\operatorname{Hom}_*(F(n),X)\to\operatorname{Hom}_*(\Sigma^tF(n),X)\to\cdots$$

are homotopy equivalences after passing to the $(d+1)$-connective covers, which proves the assertion directly by passing to the colimit. We claim the identity component of the homotopy fiber, that is, the identity component of $\operatorname{Hom}_*(\operatorname{cof}(v_n),X)$ is $(d-1)$-truncated. To prove this, note that

$$\Omega^{d-1}\operatorname{Hom}_*(\operatorname{cof}(v_n),X)\simeq\operatorname{Hom}_*(\Sigma^{d-1}\operatorname{cof}(v_n),X)\simeq *$$

where the last equivalence is by the fact that $X$ is $\Sigma^{d-1}\operatorname{cof}(v_n)$-null, since type $\Sigma^{d-1}\operatorname{cof}(v_n)\geq n+1$, and conn $\Sigma^{d-1}\operatorname{cof}(v_n)\geq d$, and by theorem 18.3.5 we have proven the result. □

**Theorem 19.1.18.** *Let $f:X\to Y$ be a map of pointed spaces. Then $L^f_{n,d}(f)$ is an equivalence iff for $0<m\leq n$, $f$ is a $v_m$-periodic equivalence and $(\pi_kX)_\mathbb{Q}\to(\pi_kY)_\mathbb{Q}$ is an equivalence for $k>d$.*

*Proof.* $\implies$: The first condition follows from theorem 19.1.16, and the second follows from the fact that $\operatorname{fib}(f)$ must be $F(n+1,d)$-acyclic, hence must be rationally trivial on degree greater than $d$ (the degrees below $d$ are being removed in the construction).

$\impliedby$: We apply induction on $n$. For the case where $n=0$, we have $L^f_0(X)\simeq(\tau_{\geq d+1}X)_\mathbb{Q}$, where the assumptions clearly imply $L^f_{n,d}(f)$ is an equivalence. For $n>0$, WLOG assume that $X$ and $Y$ are already $(d+1)$-connected, $p$-local and $F(n+1,d)$-null. Then by the assumption the fiber $F=\operatorname{fib}(f)$ is rationally trivial and $v_m$-periodic homotopy equivalent to a point for $0<m\leq n$. Let $F(n,d')$ be a type $n$ space of connectivity $d'$ equipped with a $v_n$-self map. By proposition 19.1.17, the space $\operatorname{Hom}_*(F(n,d'),F)$ is $d$-truncated since it is $v_n$-homotopy equivalent to a point. By replacing $F(n,d')$ by a suitable suspension we may assume $\operatorname{Hom}_*(F(n,d'),F)\simeq *$. We moreover assume that $d'>d$. Therefore $F$ is $F(n,d')$-null. Then $\tau_{\geq d'+1}F$ is also $F(n,d')$-null since $\tau_{\geq d'+1}$ is a right adjoint, hence $\tau_{\geq d'+1}F$ can be regarded as an object in $L^f_{n-1}(\mathbf{An}_*)^{>d'}_{(p)}$. By the induction hypothesis on $n$, we conclude that $\tau_{\geq d'+1}F\simeq *$. Moreover, by assumption $\tau_{\geq d+1}F$ is rationally trivial and $F(n+1,d)$-null, by proposition 19.1.3 we have $P_{F(n+1,d)}\tau_{\geq d'+1}F\simeq\tau_{\geq d+1}F$. Therefore we have $F$ is $d$-truncated. Moreover, since it is a fiber of a map of $d+1$-connective spaces, it must be $d$-connective. Therefore $F$ is concentrated at degree $d$, and hence $F \simeq K(G,d)$ for some $G$. Since $f$ is a rational equivalence on degree greater than $d$, $G$ must be torsion, and since $F$ is $p$-local, $G$ must be $p$-power torsion. Thus $K(G,d)$ is $F(n+1,d)$-periodic by lemma 19.1.4. Thus $f$ is an equivalence since $X$ and $Y$ are $F(n+1,d)$-null. □

**Corollary 19.1.19.** *The functor $L^f_{n,d} : \mathbf{An}_* \to L^f_n(\mathbf{An}_*)^{>d}_{(p)}$ exhibits the category $L^f_n(\mathbf{An}_*)^{>d}_{(p)}$ as the localization of $\mathbf{An}_*$ with respect to the maps satisfying the conditions in theorem 19.1.18.*

*Proof.* Directly by the definition of localizations and theorem 19.1.18. □

**Lemma 19.1.20.** *For a $F(n)$-null space $X$, we have $\Phi_{F(m)}(X) \simeq 0$ for all $m > n$.*

*Proof.* By replacing $F(m)$ with a suspension, we may assume that the connectivity of $F(m)$ is greater than the connectivity of $F(n)$. Therefore by theorem 18.3.5 we have $X$ is $F(m)$-null. Therefore the mapping space $\mathrm{Hom}_*(F(m), X) \simeq *$, and hence $\Phi_{F(m)}(X) \simeq 0$. □

**Definition 19.1.21.** We let $\mathbf{An}^{v_n}_*$ be the full subcategory of $L^f_n(\mathbf{An}_*)^{>d}_{(p)}$ consisting of $F(n,d)$-periodic objects. Explicitly, $\mathbf{An}^{v_n}_*$ is the category of spaces that are $(d+1)$-connective, $p$-local, $F(n+1,d)$-null, and $F(n,d)$-periodic.

We define a localization functor $M^f_n : \mathbf{An}_* \to \mathbf{An}^{v_n}_*$ by

$$X \mapsto \tau_{\geq d+1} \operatorname{fib}(P_{F(n+1,d)}(\tau_{\geq d+1} X_{(p)}) \to P_{F(n,d)}(\tau_{\geq d+1} X_{(p)}))$$

**Corollary 19.1.22.** *A morphism in $\mathbf{An}^{v_n}_*$ is an equivalence iff it is an $v_n$-periodic homotopy equivalence.*

*Proof.* By theorem 19.1.18. □

**Corollary 19.1.23.** *The functor $M^f_n : \mathbf{An}_* \to \mathbf{An}^{v_n}_*$ exhibits $\mathbf{An}^{v_n}_*$ as the localization of $\mathbf{An}_*$ with respect to the $v_n$-periodic equivalences.*

*Proof.* Directly by corollary 19.1.22 □

**Corollary 19.1.24.** *The stabilization $\mathbf{Sp}(\mathbf{An}^{v_n}_*)$ can be identified with $\mathbf{Sp}_{T(n)}$. We denote the associated suspension spectrum and infinite loopspace functor by $\Sigma^\infty_{T(n)}$ and $\Omega^\infty_{T(n)}$. Moreover, we have $\Sigma^\infty_{T(n)} \simeq L_{T(n)}\Sigma^\infty$ (under a certain inclusion $\mathbf{An}^{v_n}_* \hookrightarrow \mathbf{An}_*$) and $\Omega^\infty_{T(n)} \simeq \Omega^\infty \circ \tau_{\geq d_{n+1}+1}$, where $d_{n+1}$ depends on the inclusion chosen $\mathbf{An}^{v_n}_* \hookrightarrow \mathbf{An}_*$.*

*Proof.* By verifying universal properties, $\mathbf{Sp}(\mathcal{C}[W^{-1}])$ can be identified with the category $\mathbf{Sp}(\mathcal{C})[\Sigma^\infty W^{-1}]$. Therefore $\mathbf{Sp}(\mathbf{An}^{v_n}_*)$ is the category obtained by inverting the $v_n$-periodic equivalences in $\mathbf{Sp}$, which is obviously $\mathbf{Sp}_{T(n)}$. □

The above corollary shows that the nature of the category $\mathbf{An}^{v_n}_*$ do not depend on the value of $d$ in $L^f_n(\mathbf{An}_*)^{>d}_{(p)}$, since $\mathbf{An}^{v_n}_*$ is uniquely determined by its universal property as a localization. However, what $d$ does affect is the inclusion functor $\mathbf{An}^{v_n}_* \hookrightarrow \mathbf{An}_*$, which must have image in $(d+1)$-connected spaces.

**Theorem 19.1.25.** *Let $F(n)$ be a type $n$ space equipped with a $v_n$-self map. Then the natural transformation $X \to M^f_n X$ induces an equivalence $\Phi_{F(n)}(X) \to \Phi_{F(n)}(M^f_n X)$.*

*Proof.* Recall in theorem 19.1.16, the natural transformations $X \to P_A X$, $X \to X_{(p)}$, and $\tau_{\geq d+1} X \to X$ are send to equivalences by $\Phi_{F(n)}$. Therefore it suffices to show that for a $F(n+1,d)$-null space $X$, the construction $X \mapsto \operatorname{fib}(X \to P_{F(n,d)} X)$ does not change the $v_n$-periodic homotopy groups of $X$, which is obvious since $P_{F(n,d)} X$ has trivial $v_n$-periodic homotopy groups. □

## 19.2 The construction of the Bousfield Kuhn functor

In the last chapter we introduced the notion $\Phi_{F(n)}(X)$, which is exactly the information of the $v_n$-periodic homotopy groups of $X$. However, this construction (and also the $v_n$-periodic homotopy groups) depend on the choice of the type $n$ $p$-local space $F(n)$. In this section, we introduce the notion of the Bousfield-Kuhn functor, which unifies the construction $\Phi_{F(n)}(X)$. Namely, we prove the following theorem.

**Theorem 19.2.1.** *There is a unique functor $\Phi : \mathbf{An}_* \to \mathbf{Sp}_{T(n)}$ such that for any finite $p$-local type $n$ space $F(n)$, we have $\Phi_{F(n)}(X) \simeq \mathrm{Map}(F(n), \Phi(X))$.*

*Proof.* We firstly notice that the notion $\Phi_{F(n)}(X)$ can be extended naturally to a functor $\Phi_-(-) : (\mathbf{Sp}_n)^{\mathrm{op}} \to \mathrm{Fun}(\mathbf{An}_*, \mathbf{Sp}_{T(n)})$, where $\mathbf{Sp}_n$ is the subcategory of $p$-local finite spectra of type $\geq n$. This is essentially by lemma 19.1.11 and corollary 11.4.18. Then we consider $F : (\mathbf{Sp}_{\mathrm{fin}})^{\mathrm{op}} \to \mathrm{Fun}(\mathbf{An}_*, \mathbf{Sp}_{T(n)})$ as the right Kan extension of $\Phi_-(-)$. Since $\mathbf{Sp}_{T(n)}$ is closed under limits, the right Kan extension is well defined. Then we let $\Phi = F(\mathbb{S}_{(p)})$, namely by the construction of the right Kan extension,

$$\Phi(X) = \varprojlim \Phi_E(X)$$

where the limit is taken on the over category $(\mathbf{Sp}_n^{\mathrm{op}})_{\mathbb{S}_{(p)}/}$. Then by lemma 19.1.13, we have $\Phi_-(-) : (\mathbf{Sp}_n)^{\mathrm{op}} \to \mathrm{Fun}(\mathbf{An}_* \to \mathbf{Sp}_{T(n)})$ is left exact. Therefore $F$ is also left exact, and since

$$F(\mathbb{S}_{(p)})(X) = \Phi(X) \simeq \mathrm{Map}(\mathbb{S}_{(p)}, \Phi(X))$$

By the definition of the Kan extension and the definition of extension by limits we have

$$F(F(n))(X) = \Phi_{F(n)}(X) = \mathrm{Map}(F(n), \Phi(X))$$

Then we prove uniqueness. Suppose $\Phi' : \mathbf{An}_* \to \mathbf{Sp}_{T(n)}$ is another functor satisfying the given conditions, then let $F' : (\mathbf{Sp}_{\mathrm{fin}})^{\mathrm{op}} \to \mathrm{Fun}(\mathbf{An}_*, \mathbf{Sp}_{T(n)})$ by letting $F'(E)(X) = \mathrm{Map}(E, \Phi'(X))$. Then by the universal property of the Kan extension we obtain a natural transformation $F' \to F$, which is an equivalence on the full subcategory spanned by spectra of type $\geq n$. Therefore for some type $n$ spectrum $F(n)$, we have $\mathrm{Map}(F(n), \Phi(X)) \simeq \mathrm{Map}(F(n), \Phi'(X))$. By taking duals we have $\mathbb{D}\, F(n) \otimes \Phi(X) \simeq \mathbb{D}\, F(n) \otimes \Phi'(X)$. By passing to the colimit we have $\Phi(X) \to \Phi'(X)$ is a $T(n)$-equivalence. But since both $\Phi(X)$ and $\Phi'(X)$ are $T(n)$-local, they must be equivalent. □

*Remark* 27. In practice, we often find a cofinal system of type $n$ spectra

$$E_0 \to E_1 \to \cdots$$

among all type $n$ spectra with a map to $\mathbb{S}_{(p)}$, and use $\varprojlim_k \Phi_{E_k}(X)$ to calculate $\Phi(X)$.

**Proposition 19.2.2.** *The functor $\Phi : \mathbf{An}_* \to \mathbf{Sp}_{T(n)}$ is left exact.*

*Proof.* It suffices to prove $\Phi_{F(n)} : \mathbf{An}_* \to \mathbf{Sp}_{T(n)}$ is left exact, where $F(n)$ is some type $n$ space with a $v_n$-self map. Since limits of spectra are computed spacewise, it suffices to prove $\Omega^{\infty+k}\Phi_{F(n)}$ is left exact. Then since $\Phi_{F(n)}$ is a periodic spectrum, it suffices to show $\Omega^\infty \Phi_{F(n)}$ is left exact. But by construction,

$$\Omega^\infty \Phi_{F(n)} X = \varinjlim(\mathrm{Hom}_*(F(n), X) \to \mathrm{Hom}_*(\Sigma^t F(n), X) \to \cdots)$$

which is clearly right exact since finite limits commutes with filtered colimits. □

**Proposition 19.2.3.** *A map $f : X \to Y$ of pointed spaces is a $v_n$-periodic homotopy equivalence iff $\Phi(f) : \Phi(X) \to \Phi(Y)$ is an equivalence.*

*Proof.* $\Longrightarrow$ is obvious since by definition $v_n$-periodic homotopy equivalences are the maps send by $\Phi_{F(n)}$ to equivalences. Therefore by passing to limits we have $\Phi(f)$ an equivalence.

$\Longleftarrow$ is also obvious since $\Phi_{F(n)}(X) \simeq \mathrm{Map}(F(n), \Phi(X))$. □

**Corollary 19.2.4.** *The functor $\Phi : \mathbf{An}_* \to \mathbf{Sp}_{T(n)}$ factors through the localization $\mathbf{An}_* \to \mathbf{An}_*^{v_n}$.*

*Proof.* Directly by corollary 19.1.22 and proposition 19.2.3. □

**Theorem 19.2.5.** *The functor $\Phi : \mathbf{An}_*^{v_n} \to \mathbf{Sp}_{T(n)}$ admits a left adjoint $\Theta : \mathbf{Sp}_{T(n)} \to \mathbf{An}_*^{v_n}$.*

*Proof.* By the construction of $\Phi$ as a right Kan extension, it suffices to prove $\Phi_{F(n)}$ for $F(n)$ a type $n$ space with a $v_n$-self map $v_n : \Sigma^t F(n) \to F(n)$ has a left adjoint $\Theta_{F(n)}$ (then we may construct $\Theta$ as a colimit of $\Theta_{F(n)}$). Explicitly, we show that for each spectrum $Z$, the functor $X \mapsto \mathrm{Hom}(Z, \Phi_{F(n)}(X))$ is corepresented by some $\Theta_{F(n)}(Z) \in \mathbf{An}_*^{v_n}$.

Since left adjoints preserves colimits, it suffices to prove $\Theta_{F(n)}(Z)$ exists for $Z = \mathbb{S}^{kt}$ with $k$ any integer (possibly negative). Using $t$-periodicity of $\Phi_{F(n)}(X)$ it suffices to prove the case for $Z = \mathbb{S}$, that is, the functor $X \mapsto \Omega^\infty \Phi_{F(n)}(X)$ is corepresentable. But by proposition 19.1.17 we have the sequence

$$\mathrm{Hom}_*(F(n), X) \to \mathrm{Hom}_*(\Sigma^t F(n), X) \to \cdots$$

is eventually constant since it is constant after taking a $d$-connective cover for a $F(n+1, d)$-null space $X$. Thus we only need to show that the functor $X \mapsto \mathrm{Hom}_*(\Sigma^{ct} F(n), X)$ is corepresentable for some $c \gg 0$ (since $X \in \mathbf{An}_*^{v_n}$, which is also why the theorem does not hold if we regard $\Phi$ as a functor $\mathbf{An}_* \to \mathbf{Sp}$).

We claim this is corepresented by $P_{F(n+1,d)}(\Sigma^{ct} F(n))$. This space obviously lies in $L_n^f(\mathbf{An}_*)_{(p)}^{>d}$, and for $c \gg 0$, $P_{F(n,d)}(\Sigma^{ct} F(n)) \simeq *$ by theorem 18.3.5 and the definition of the unstable Bousfield class. Hence $P_{F(n+1,d)}(\Sigma^{ct} F(n)) \in \mathbf{An}_*^{v_n}$, and represents the functor, which proves the theorem. □

The most fundamental property of the Bousfield Kuhn functor is that the $T(n)$-localization functor of spectra factors through it.

**Theorem 19.2.6.** *Let $X$ be a spectrum. Then we have a canonical equivalence $\Phi\Omega^\infty(X) \simeq L_{T(n)}(X)$.*

*Proof.* We first calculate $\Phi_{F(n)}\Omega^\infty(X)$ where $F(n)$ is a type $n$ space equipped with a $v_n$-self map. We have

$$\begin{aligned}
&\Omega^\infty \Phi_{F(n)}(\Omega^\infty X) \\
\simeq &\varinjlim(\mathrm{Hom}_*(F(n), \Omega^\infty X) \to \mathrm{Hom}_*(\Sigma^t F(n), \Omega^\infty X) \to \cdots) \\
\simeq &\varinjlim(\mathrm{Hom}(\Sigma^\infty F(n), X) \to \mathrm{Hom}(\Sigma^{\infty+t} F(n), X) \to \cdots) \\
\simeq &\varinjlim(\Omega^\infty D\Sigma^\infty F(n) \otimes X \to \Omega^{\infty-t} D\Sigma^\infty F(n) \otimes X \to \cdots) \\
\simeq &\Omega^\infty \mathbb{D} F(n)[v_n^{-1}] \otimes X \\
\simeq &\Omega^\infty \mathbb{D} F(n)[v_n^{-1}] \otimes L_{T(n)} X
\end{aligned}$$

where the fourth equivalence is since $\Omega^\infty$, as a right adjoint, commutes with filtered colimits, and the final equivalence is since $\mathbb{D} F(n)[v_n^{-1}]$ is a telescope of type $n$, hence $\mathbb{D} F(n)[v_n^{-1}]$ calculates the $T(n)$-homology of $X$ for some $X$, hence it is safe to replace $X$ by $L_{T(n)}(X)$. However, when $X$ is itself $T(n)$-local, the sequence

$$\mathrm{Hom}(\Sigma^\infty F(n), X) \to \mathrm{Hom}(\Sigma^{\infty+t} F(n), X) \to \cdots$$

is actually constant since the fiber of the map is $\operatorname{Hom}(\Sigma^\infty F(n)/v_n, X)$, and the previous spectrum is a spectrum of type $n+1$, and since $X$ is $T(n)$-local hence $F(n+1)$-null, this fiber is null. Therefore we have

$$\Omega^\infty \Phi_{F(n)}(\Omega^\infty X) \simeq \Omega^\infty \operatorname{Map}(\Sigma^\infty F(n), L_{T(n)}X)$$

and since both sides are periodic, we have $\Phi_{F(n)}(\Omega^\infty X) \simeq \operatorname{Map}(\Sigma^\infty F(n), L_{T(n)}X)$. This clearly holds after replacing $\Sigma^\infty F(n)$ by any type $n$ spectrum.

Thus the conclusion follows from

$$\Phi(\Omega^\infty X) \simeq \varprojlim_E \Phi_E(\Omega^\infty X) \simeq \varprojlim_E \operatorname{Map}(E, L_{T(n)}X) \simeq L_{T(n)}X$$

$\square$

**Corollary 19.2.7.** *We have two pairs of adjunctions*

$$\begin{tikzcd} \mathbf{Sp}_{T(n)} \arrow[r, shift left, "\Theta"] & \mathbf{An}_*^{v_n} \arrow[l, shift left, "\Phi"] \arrow[r, shift left, "\Sigma^\infty_{T(n)}"] & \mathbf{Sp}_{T(n)} \arrow[l, shift left, "\Omega^\infty_{T(n)}"] \end{tikzcd}$$

*such that the horizontal compositions are the identities.*

*Proof.* theorem 19.2.6 implies that the bottom composition is the identity, and the top row is left adjoint to the bottom row, hence the identity. $\square$

## 19.3 Monadicity of the Bousfield Kuhn functor

We first recall a few facts on monads.

**Definition 19.3.1.** Recall that traditionally (or in the 1-categorical sense), a monad on a 1-category $\mathcal{C}$ is an endofunctor $T : \mathcal{C} \to \mathcal{C}$ equipped with a natural transformation $\mu : T \circ T \to T$ and a natural transformtion $\eta : \mathrm{id}_{\mathcal{C}} \to T$ such that the following diagrams commute.

$$\begin{tikzcd} T\circ T\circ T \arrow[r, "\mu\circ 1_T"] \arrow[d, "1_T\circ\mu"'] & T\circ T \arrow[d, "\mu"] \\ T\circ T \arrow[r, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "1_T\circ\eta"] \arrow[dr, "1_T"'] & T\circ T \arrow[d, "\mu"] & T \arrow[l, "\eta\circ 1_T"'] \arrow[dl, "1_T"] \\ & T & \end{tikzcd}$$

where $1_T$ denotes the trivial natural transformation $1_T : T \to T$ given by id. Finally, an algebra of a monad is an object of $\mathcal{C}$, say $X$, equipped with a morphism $m : TX \to X$ such that the following diagrams commute

$$\begin{tikzcd} T\circ TX \arrow[r, "\mu\circ 1_X"] \arrow[d, "1_T\circ m"'] & TX \arrow[d, "m"] \\ TX \arrow[r, "m"'] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "1_X"'] & TX \arrow[d, "m"] \\ & X \end{tikzcd}$$

where $1_X$ is the identity map on $X$.

The constructions immediately generalizes to $\infty$-categories. Note that the above definitions abstracts to the following. Let $\mathcal{C}$ be an $\infty$-category, then $\operatorname{Fun}(\mathcal{C},\mathcal{C}) \simeq \operatorname{Hom}_{\mathbf{Cat}}(\mathcal{C},\mathcal{C})$ has a tautological structure of an $\mathbb{E}_1$ monoidal category with tensor product given by composition of functors (here **Cat** is regarded as a $(\infty,2)$-category). Then $\mathcal{C}$ is a category left tensored over $\operatorname{Fun}(\mathcal{C},\mathcal{C})$ given by applying an endofunctor on an object. Therefore, the above definition of monads generalizes as follows. We define a monad $T$ on a category $\mathcal{C}$ to be an $\mathbb{E}_1$-algebra in $\operatorname{Fun}(\mathcal{C},\mathcal{C})$, and a $T$-algebra $X$ to be an object of $\mathbf{LMod}_T(\mathcal{C})$.

**Construction 19.3.2.** For a monad $T$ on $\mathcal{C}$, notice that we have the following adjunction

$$T : \mathcal{C} \rightleftarrows \mathbf{LMod}_T(\mathcal{C}) : U$$

which also recovers the monad by the composition $UT$. For a pair of adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$, we have $GF$ a monad on $\mathcal{C}$, with its category of algebras equivalent to the essential image of $G$.

Explicitly, the $\mathbb{E}_1$-algebra structure of $GF$ has the multiplication map $1_G \circ \varepsilon \circ 1_F : GFGF \to GF$ and the unit map $\eta : \mathrm{id}_{\mathcal{C}} \to GF$. Furthermore, notice that for any element $Y \in \mathcal{D}$, $GY$ is a $GF$-algebra with the action given by $1_G \circ \varepsilon \circ 1_Y : GFGY \to GY$. Hence $G$ upgrades to a comparison functor (the Eilenberg-Moore functor) $E : \mathcal{D} \to \mathbf{LMod}_{GF}(\mathcal{C})$, and we say $GF$ is monadic iff the previous comparison functor is an equivalence.

We have the further definition of the Kleisli category, or the category of free $GF$-modules $\mathbf{LMod}^{\mathrm{free}}_{GF}(\mathcal{C})$, defined to be the full subcategory of $\mathbf{LMod}_{GF}(\mathcal{C})$ consisting of the $GF$-modules of the form $GFX$ equipped with the natural module structure. By the definition of adjunctions, a morphism of $GF$-modules $GFX \to GFY$ is equivalent to the data of a morphism $X \to GFY$ as objects of $\mathcal{C}$. Therefore, the Kleisli category is equivalent to a category having the same objects as $\mathcal{C}$, and a morphism $X \to Y$ is defined to be a morphism $X \to GFY$ in $\mathcal{C}$ (this reduces our construction to the usual construction of Kleisli categories). This allows us to define a map

$$K : \mathbf{LMod}^{\mathrm{free}}_{GF}(\mathcal{C}) \to \mathcal{D}$$

given by sending an object $X \in \mathbf{LMod}^{\mathrm{free}}_{GF}(\mathcal{C})$ via the second construction as an object in $\mathcal{C}$ to $FX$, and a morphism $X \to Y$ regarded as a morphism $X \to GFY$ is send to $FX \to FY$ given by the defintion of adjunctions. By this identification we have obviously that $K$ is fully faithful.

In the aobve constructions, we see that we have $E \circ K : \mathbf{LMod}^{\mathrm{free}}_{GF}(\mathcal{C}) \to \mathbf{LMod}_{GF}(\mathcal{C})$ is naturally equivalent to the inclusion functor of the free modules into all modules. In particular, $E \circ K$ is fully faithful.

We have the following fundamental theorem of Barr-Beck on the monadicity of adjoint functors, generalized by Lurie to the $\infty$-categorical case, whose proof we will omit.

**Theorem 19.3.3** (Barr-Beck-Lurie). *The adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is monadic iff the two following conditions holds.*

1. *$G$ is conservative.*

2. *For a simplicial object $V$ of $\mathcal{D}$, if it is send to a split simplicial object in $\mathcal{C}$ by $G$, then it admit a colimit in $\mathcal{D}$, and this colimit is preserved by $G$.*

*In particular, if $\mathcal{D}$ admits all geometric realizations, all of which is preserved by $G$, then the second condition clearly holds.*

We can now state the first main result of this review.

**Theorem 19.3.4** (Eldred-Heuts-Mathew-Meier). *The adjunction $\Theta : \mathbf{Sp}_{T(n)} \rightleftarrows \mathbf{An}^{v_n}_* : \Phi$ is monadic.*

Before this, we need some lemmas on geometric realizations.

**Lemma 19.3.5.** *For a diagram of simplicial spaces $X_\bullet \to B_\bullet \leftarrow Y_\bullet$ with $B_n$ is connected for all $n$, then the natural map $|X_\bullet \times_{B_\bullet} Y_\bullet| \to |X_\bullet| \times_{|B_\bullet|} |Y_\bullet|$ is an equivalence.*

*Proof.* This is a famous result called the Bousfield-Friedlander criterion in simplicial homotopy theory, hence we will omit the proof here. Reference! □

**Lemma 19.3.6.** *Let $V \in \mathbf{An}_*$ be a finite $d$-dimensional CW complex. Then the functor $\mathrm{Hom}_*(V, -) : \mathbf{An}^{\geq d}_* \to \mathbf{An}_*$ preserves geometric realizations.*

*Proof.* Consider a simplicial space $X_\bullet$ with each $X_n$ $d$-connective. Then we consider the collection $T$ of all finite spaces $V$ such that $|\operatorname{Hom}_*(V, X_\bullet)| \simeq \operatorname{Hom}_*(V, |X_\bullet|)$. Note that the category $\mathbf{An}_*$ is a topos, and hence $T$ is closed under coproducts (wedge sums).

We apply induction to show all spaces of dimension $\leq d$ is in $T$. Firstly by definition $S^0 \in T$, and since $T$ is closed under wedge sums, 0-dimensional spaces are all in $T$. Assume now that for $n \leq d-1$ that all $n$-dimensional spaces are in $T$, then for any space $Y$ of dimension $n+1$, we have a cofiber sequence $\bigvee S^n \to \tau_{\leq n} Y \to Y$. Moreover, since $n \leq d-1$, $\operatorname{Hom}_*(S^n, X_\bullet)$ is connected, and hence by lemma 19.3.5 and that $\bigvee S^n$ and $\tau_{\leq n} Y$ are in $T$, we have $Y \in T$. $\square$

*Proof.* We now apply theorem 19.3.3 to prove theorem 19.3.4. Firstly $\Phi$ is conservative by proposition 19.2.3 and corollary 19.1.22. Hence it suffices to prove that $\Phi$ preserves geometric realizations.

First note that the colimit in $\mathbf{Sp}_{T(n)}$ is obtained by first taking the colimit in $\mathbf{Sp}$ then take the $T(n)$-localization. This allows us to regard $\Phi$ as a functor $\mathbf{An}_*^{v_n} \to \mathbf{Sp}$. Let $X_\bullet$ be a simplicial object in $\mathbf{An}_*^{v_n}$, we prove $|\Phi(X_\bullet)| \to \Phi(|X_\bullet|)$ is a $T(n)$-equivalence. Since $T(n)$ can be obtained by inverting the $v_n$-self map of any type $n$ spectrum, and the dual of a type $n$ spectrum is still type $n$, it suffices to prove

$$\operatorname{Map}(F(n), |\Phi(X_\bullet)|) \to \operatorname{Map}(F(n), \Phi(|X_\bullet|))$$

is an equivalence. Now since in the category of spectra fiber sequences coincide with the cofiber sequences, the fact that geometric realization is a colimit and hence preserves every colimit implies that it commutes with finite limits. Now $\operatorname{Map}(F(n), -)$ is a finite limit since $F(n)$ is finite, hence the left hand side can be identified with $|\operatorname{Map}(F(n), \Phi(X_\bullet))|$. Therefore it suffices to prove

$$|\operatorname{Map}(F(n), \Phi(X_\bullet))| to \operatorname{Map}(F(n), \Phi(|X_\bullet|))$$

is an equivalence. By theorem 19.2.1 and choosing $F(n)$ to be of the form $\Sigma^\infty F(n)$ for a type $n$ space $F(n)$, it suffices to prove

$$|\Phi_{F(n)}(X_\bullet)| \to \Phi_{F(n)}(|X_\bullet|)$$

is an equivalence. Then, notice that the subcategory $\mathbf{An}_*^{v_n} \subseteq L_n^f(\mathbf{An}_*)_{(p)}^{>d}$ is closed under colimits, and the colimits in $L_n^f(\mathbf{An}_*)_{(p)}^{>d}$ are computed by first taking the colimits in $\mathbf{An}_*$ and then take the localization $L_{n,d}^f$. Finally we notice that the map $L_{n,d}^f : (\mathbf{An}_*)_{(p)}^{>d} \to L_n^f(\mathbf{An}_*)_{(p)}^{>d}$ induces an equivalence on $v_n$-periodic homotopy, hence induces an equivalence after applying $\Phi_{F(n)}$. Therefore it suffices to show that $\Phi_{F(n)} : (\mathbf{An}_*)_{(p)}^{>d} \to \mathbf{Sp}$ preserves geometric realizations for some chosen $d$.

Finally we expand the definition of $\Phi_{F(n)}(X_\bullet)$ for $X_n \in (\mathbf{An}_*)_{(p)}^{>d}$. Suppose that the $v_n$-self map of $F(n)$ is $v_n : \Sigma^t F(n) \to F(n)$, we notice that

$$\Phi_{F(n)}(X_\bullet) \simeq \varinjlim(\tau_{\geq 0}\Phi_{F(n)}(X_\bullet) \to \Omega^t \tau_{\geq 0}\Phi_{F(n)}(X_\bullet) \to \cdots)$$

hence it suffices to see that $\tau_{\geq 0}\Phi_{F(n)}$ preserves geometric realizations. But note that the category of $\mathbb{E}_\infty$-groups are closed under filtered colimits as the subcategory of $\mathbf{An}_*$, hence under the identification of the category of connective spectra and $\mathbb{E}_\infty$-groups, and the fact that

$$\tau_{\geq 0}\Phi_{F(n)}(X_\bullet) \simeq B^\infty \varinjlim(\operatorname{Hom}_*(F(n), X_\bullet) \to \operatorname{Hom}_*(\Sigma^t F(n), X_\bullet) \to \cdots)$$

it suffices to show

$$|\operatorname{Hom}_*(F(n), X_\bullet)| to \operatorname{Hom}_*(F(n), |X_\bullet|)$$

is an isomorphism. But since the category $\mathbf{An}_*^{v_n}$ is independent of the choice of $d$ in $L_n^f(\mathbf{An}_*)_{(p)}^{>d}$, we may choose $d \gg 0$ such that $d > \dim F(n)$, and hence the theorem follows from lemma 19.3.6 $\square$

As a direct corollary, $\Theta$ computes the free algebra (of the monad $\Phi\Theta$) of a $T(n)$-local spectra. Therefore the computation of the functor $\Theta$ may appear as an important question. In fact, for a $T(n)$-local finite spectrum, we have the following way of computing $\Theta$.

**Proposition 19.3.7.** *Let $F(n)$ be a $(d-2)$-connected finite space of type $n$ with a $v_n$-self map $v : \Sigma^t F(n) \to F(n)$, then we have a canonical equivalence*

$$\Theta(L_{T(n)}\Sigma^{\infty+2}F(n)) \simeq L^f_{n,d}\Sigma^2 F(n)$$

*Proof.* Consider the space $F(n)/v$. Then by theorem 18.3.5, $\Sigma^i F(n)/v$ a suspension space having type $n+1$ and connectivity $d+i-1$, hence it is $F(n,d)$-periodic. Thus

$$\mathrm{Hom}_*(L^f_{n,d}\Sigma^i F(n)/v, X) \simeq \mathrm{Hom}_*(\Sigma^i F(n)/v, X) \simeq *$$

Then consider the cofiber sequence

$$\Sigma F(n)/v \to \Sigma^{t+2}F(n) \to \Sigma^t F(n) \to \Sigma^2 F(n)/v$$

Let $X \in \mathbf{An}_*^{v_n}$. By the above, we have

$$\mathrm{Hom}_*(\Sigma^2 F(n), X) \simeq \mathrm{Hom}_*(\Sigma^{t+2}F(n), X) \simeq \cdots$$

Thus for any $X \in \mathbf{An}_*^{v_n}$, we have

$$\begin{aligned}
&\mathrm{Hom}_*(\Theta(L_{T(n)}\Sigma^{\infty+2}F(n)), X)\\
\simeq\,&\mathrm{Hom}(L_{T(n)}\Sigma^{\infty+2}F(n), \Phi(X))\\
\simeq\,&\Omega^\infty\Phi_{\Sigma^2F(n)}(X)\\
\simeq\,&\mathrm{Hom}_*(\Sigma^2 F(n), X)\\
\simeq\,&\mathrm{Hom}_*(L^f_{n,d}\Sigma^2 F(n), X)
\end{aligned}$$

□

## 19.4 Calculus and dual calculus of functors

Recall that we have the general philosophy that there is a monoidal (not generally fully faithful) embedding $\mathrm{Fun}(\mathrm{Core}\,\mathbf{Fin}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{C})$ given by

$$C \mapsto (X \mapsto \bigoplus (C(n) \otimes X^{\otimes n})_{hS_n})$$

for any category $\mathcal{C}$. Thus given an operad in the classical sense, that is, a symmetric sequence $C \in \mathrm{Fun}(\mathrm{Core}\,\mathbf{Fin}, \mathcal{C})$ which is a $\mathbb{E}_1$-monoid under the composition product, it gives rise to a monad. The essense of (dual) Goodwillie calculus which is used here finds, in some sense, adjoints to this construction. In this section, $\mathcal{C}$ and $\mathcal{D}$ are *pointed* (or even stable) categories good enough so that they have all limits and colimits that are possibly implicitly needed in the conditions.

**Definition 19.4.1.** For an $S$-cube $X : P(S) \to \mathcal{C}$, we define its total fiber

$$\mathrm{tfib}\, X = \mathrm{fib}(X(\varnothing) \to \varprojlim_{T\subseteq S, T\neq\varnothing} X(T))$$

and dually its total cofiber

$$\mathrm{tcof}\, X = \mathrm{cof}(\varinjlim_{T\subseteq S, T\neq S} X(T) \to X(S))$$

**Corollary 19.4.2.** *The construction of total (co)fibers are iterative. That is, given a cube $X : P(S) \to \mathcal{C}$ and $s \in S$, we have* $\mathrm{tfib}(X)$ *identified with the total fiber of the following cube*

$$Y : P(S - \{s\}) \to \mathcal{C}, Y(T) = \mathrm{fib}(Y(T) \to Y(T \cup \{s\}))$$

*The dual statement holds for the total cofiber.*

*Proof.* By directly verifying the universal property. □

**Definition 19.4.3.** Consider a finite set $S$, and let $P(S)$ be the poset of subsetes of $S$ ordered by inclusion. We define $P_{\leq i}(S)$ to be the subset of $P(S)$ consisting of subsets having cardinality smaller or equal than $i$ and $P_{>i}(S)$ to be the subset of $P(S)$ consisting of subsets having cardinality greater than $i$.

We define an $S$-cube $F : P(S) \to \mathcal{C}$ (where $S$ is a finite set) to be strongly coCartesian if it is the left Kan extension of the subdiagram $F : P_{\leq 1}(S) \to \mathcal{C}$. We define a cube to be Cartesian if it is the right Kan extension of the subdiagram $F : P_{>0}(S) \to \mathcal{C}$.

We define a functor $F : \mathcal{C} \to \mathcal{D}$ to be $n$-excisive if it takes strongly coCartesian $[n]$-cubes to Cartesian $[n]$-cubes (here $[n] = \{0, 1, \cdots, n\}$). Clearly we have a functor is $n$-excisive implies that it is $n+1$-excisive. We define the full subcategory $\mathrm{Exc}_n(\mathcal{C}, \mathcal{D})$ to be the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the $n$-excisive functors.

Also, we form the dual of the above construction. We have the dual definition of strongly Cartesian cubes and coCartesian cubes. We define a functor to be $n$-coexcisive if it takes strongly Cartesian $[n]$-cubes to coCartesian $[n]$-cubes. Dually we also have a functor is $n$-coexcisive implies that it is $n+1$-excisive. We define the category of coexcisive functors to be $\mathrm{Exc}^n(\mathcal{C}, \mathcal{D})$.

Note that when $\mathcal{C}$ and $\mathcal{D}$ are both stable, the notion of excisive and coexcisive functors coincide.

As we are working with functors $F : \mathbf{Sp}_{T(n)} \to \mathbf{Sp}_{T(n)}$, we hence assume from now on $\mathcal{C}$ and $\mathcal{D}$ are both stable (even some arguments still work when they are not), so that the excisive functors are exactly the coexcisive functors.

**Construction 19.4.4.** We now introduce methods to detect whether a functor is $n$-excisive. We define the $n$-th cross effect of a functor $F : \mathcal{C} \to \mathcal{D}$ to be the functor $\mathcal{C}^n \to \mathcal{D}$ given by

$$\mathrm{cr}_n(F)(X_1, \dots, X_n) = \mathrm{tfib}_{T \subseteq S}(F(\coprod_{s \notin T} X_s))$$

where $S = \{1, \dots, n\}$.

Dually we define the cocross effect to be

$$\mathrm{cr}^n(F)(X_1, \dots, X_n) = \mathrm{tcof}_{T \subseteq S}(F(\prod_{s \in T} X_s))$$

The two functors detect whether $F$ sends strongly coCartesian $S$-cubes to Cartesian $S$-cubes (or its dual version), hence gives the obvious criterion that $F$ is $n$-excisive iff $\mathrm{cr}_{n+1}(F) = 0$. Note that the construction of (co)cross effects can be iterative, that is, we have

$$\mathrm{cr}_n(F)(X_1, \dots, X_n) \simeq \mathrm{cr}_2(\mathrm{cr}_{n-1}(F)(X_1, \dots, X_{n-2}, -))(X_{n-1}, X_n)$$

and

$$\mathrm{cr}^n(F)(X_1, \dots, X_n) \simeq \mathrm{cr}^2(\mathrm{cr}^{n-1}(F)(X_1, \dots, X_{n-2}, -))(X_{n-1}, X_n)$$

This follows from the iterative construction of the total (co)fibers.

**Construction 19.4.5.** We note that taking the cross effect of a functor $F : \mathcal{C} \to \mathcal{D}$ defines a functor

$$: \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

given by

$$\mathrm{cr}_n^{\Delta}(F)(X) = \mathrm{cr}_n(F)(X, \dots, X)$$

Since $\mathrm{cr}_n(X_1, \dots, X_n)$ can be written as a retract of $\mathrm{cr}_n(\coprod X_i, \dots, \coprod X_i)$, we have $F$ $n$-excisive iff $\mathrm{cr}_{n+1}^{\Delta}(F) \simeq *$. Noticing that $F \simeq \mathrm{cr}_1^{\Delta}(F)$, we have a natural transformation $\varepsilon : \mathrm{cr}_n^{\Delta}(F) \to F$ given by

$$\mathrm{cr}_n \Delta(F)(X) \simeq \mathrm{cr}_n(F)(X, \dots, X) \to F(\coprod X) \to F(X)$$

where the second map is given by the structure map of the total fiber. We define the cofiber of this natural tranformation to be $G_n F = \mathrm{cof}(\varepsilon)$, and define

$$P_n F = \varinjlim(F \to G_{n+1}F \to G_{n+1}G_{n+1}F \to \cdots)$$

By checking definitions we see that $P_n F$ is the couniversal $n$-excisive functor equipped with a natural transformation $F \to P_n F$. In other terms, we have $P_n$ a localization functor left adjoint to the inclusion $\mathrm{Exc}_n(\mathcal{C}, \mathcal{D}) \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Dually we define

$$\mathrm{cr}^n_{\Delta}(F)(X) = \mathrm{cr}^n(F)(X, \dots, X)$$

and a functor is $n$-coexcisive iff $\mathrm{cr}^n_{\Delta}(F) \simeq *$. Also we have a natural transformation $\eta : F \to \mathrm{cr}^n_{\Delta}(F)$ given by

$$F(X) \to F(\prod X) \to \mathrm{cr}^n(F)(X, \dots, X) \simeq \mathrm{cr}^n_{\Delta}(F)(X)$$

take $G^n F = \mathrm{fib}(\eta)$, and finally

$$P^n F = \varprojlim(F \leftarrow G^{n+1}F \leftarrow \cdots)$$

Again we have $P^n F$ is the universal $n$-coexcisive functor equipped with a natural transformation $P^n F \to F$. $P^n$ is also the colocalization functor right adjoint to the inclusion $\mathrm{Exc}^n(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$. For a functor $F$, we define its $n$-homogeneous part $D_n F = \mathrm{fib}(P_n F \to P_{n-1}F)$. Dually its $n$-cohomogeneous part $D^n F = \mathrm{cof}(P^{n-1}F \to P^n F)$.

**Definition 19.4.6.** We define a functor $F : \mathcal{C}^n \to \mathcal{D}$ to be multilinear if it is 1-excisive in each variable. Dually we have the definition of comultilinear functors.

**Proposition 19.4.7.** *If $F : \mathcal{C} \to \mathcal{D}$ is $n$-excisive, then for $0 \le m \le n$ we have the functor $\mathrm{cr}_{m+1}(F)$ is $(n-m)$-excisive in each variable.*

*Proof.* By the iterative construction of cross effects in definition 19.4.1, we have

$$\mathrm{cr}_{m+1}(F)(X_1, \dots, X_n, Y) \simeq \mathrm{cr}_m(F(- \coprod Y))(X_1, \dots, X_n)$$

and hence by induction it suffices to prove that if $F$ is $n$-excisive, then $F(- \coprod Y)$ is $(n-1)$-excisive, which is obvious. ☐

**Theorem 19.4.8.** *We have an equivalence of categories between the category of $n$-homogeneous functors $\mathcal{C} \to \mathcal{D}$ with the category of symmetric multilinear functors $\mathcal{C}^n \to \mathcal{D}$, with the equivalence given by $F(X) \mapsto \mathrm{cr}_n(F)(X_1, \dots, X_n)$ and $G(X_1, \dots, X_n) \mapsto G(X, \dots, X)_{hS_n}$. We define this second operation by $\Delta_{S_n}$.*

*Dually, we have an equivalence of categories of $n$-cohomogeneous functors with the category of symmetric multilinear functors, with the equivalence given by $F(X) \mapsto \mathrm{cr}^n(F)(X_1, \dots, X_n)$ and $G(X_1, \dots, X_n) \mapsto G(X, \dots, X)^{hS_n}$. We define this by $\Delta^{S_n}$.*

*Remark* 28. Note that in the stable settings, the $n$-excisive and coexcisive functors agree with each other, and the difference of $P_n$ and $P^n$ comes froms the fact that they are respectively the left and right adjoint to the inclusion. Here, the difference is similar.

*Proof.* We first verify that for a symmetric multilinear functor $L$ we have $\mathrm{cr}_n \Delta_{S_n} L \to L$ is an isomorphism. By unwinding definitions, we have

$$\begin{aligned}
&\mathrm{cr}_n \Delta_{S_n} L(X_1, \dots, X_n) \\
\simeq &\,\mathrm{tfib}(T \mapsto L(\coprod_{s \notin T} X_s, \dots, \coprod_{s \notin T} X_s)_{hS_n}) \\
\simeq &\,\mathrm{tfib}(T \mapsto L(\coprod_{s \notin T} X_s, \dots, \coprod_{s \notin T} X_s))_{hS_n} (\text{by stability})
\end{aligned}$$

Since $L$ is 1-excisive in each factor, we have

$$L(\coprod_{s \notin T} X_s, \dots, \coprod_{s \notin T} X_s) \simeq \prod_{f \in T^n} L(X_{f(1)}, \dots, X_{f(n)})$$

where $T \subseteq S$, $S = \{1, \dots, n\}$. If we denote this cube by $\mathcal{X}$, then this cube can be decomposed into the following. Let $\sigma : S \to S$ be an arbitrary map, and let $\mathcal{X}_\sigma$ denote the cube where $\mathcal{X}_\sigma(T) = L(X_{\sigma(1)}, \dots, X_{\sigma(n)})$ if $\sigma(S) \subseteq S - T$ and is trivial otherwise. Then we clearly have $\mathcal{X} \simeq \prod_{\sigma : S \to S} \mathcal{X}_\sigma$.

Then note that if $\sigma$ is not a bijection we have $\mathrm{tfib}(\mathcal{X}_\sigma)$, then we have $\mathrm{tfib}(\mathcal{X}_\sigma)$ is trivial since in this cube, for $s \notin \mathrm{im}(\sigma)$, the morphism pointing from $T \to T \cup \{s\}$ is an isomorphism, and by the iterative construction of total fibers we have the total fiber is trivial by first taking fibers in this direction.

Finally for $\sigma$ is indeed a bijection, we have $\mathcal{X}_\sigma$ is trivial except for the initial vertex with value $L(X_{\sigma(1)}, \dots, X_{\sigma(n)})$, hence we have

$$\mathrm{tfib}(\mathcal{X}_\sigma) \simeq L(X_{\sigma(1)}, \dots, X_{\sigma(n)})$$

Therefore

$$\mathrm{tfib}(\mathcal{X}) \simeq \prod_{\sigma \in S_n} L(X_{\sigma(1)}, \dots, X_{\sigma(n)})$$

and the map $\mathrm{cr}_n \Delta_{S_n} L \to L$ is tautological.

Since $L$ is symmetric, we have

$$\mathrm{cr}_n \Delta_{S_n} L(X_1, \dots, X_n) \simeq (\prod_{\sigma \in S_n} L(X_{\sigma(1)}, \dots, X_{\sigma(n)}))_{hS_n} \simeq L(X_1, \dots, X_n)$$

Then we treat the converse case. For an $n$-homogeneous functor $F$, note that $\mathrm{cr}_n(F) \simeq *$ implies that $F$ is also trivial, since $F$ is also $(n-1)$-excisive by definition. Therefore to prove that $\Delta_{S_n} \mathrm{cr}_n(F) \to F$ is an equivalence of $n$-homogeneous functors, it suffices to prove $\mathrm{cr}_n \Delta_{S_n} \mathrm{cr}_n(F) \to \mathrm{cr}_n(F)$ is an equivalence of symmetric multilinear functors.

We first construct the natural transformation $\gamma : \Delta_{S_n} \mathrm{cr}_n(F) \to F$. We define

$$\begin{aligned}
\gamma_X : &\Delta_{S_n} \mathrm{cr}_n(F)(X) \\
&\simeq (\mathrm{cr}_n(F))(X, \dots, X)_{hS_n} \\
&\to F(X \amalg \cdots \amalg X)_{hS_n} \\
&\to F(X)_{hS_n} \\
&\to F(X)
\end{aligned}$$

where the first arrow is given by the structure map of the total fiber, and we have the last arrow since $F(X)$ has the trivial $S_n$-action. Then note that this map is an equivalence after passing to $\mathrm{cr}_n$ since it agrees with the equivalence defined above, we have the result. $\square$

**Corollary 19.4.9.** *We have*

$$D_n(F) \simeq \varinjlim_m \Omega^{mn}(\mathrm{cr}_n(F))(\Sigma^m X, \dots, \Sigma^m X)$$

*Dually we have*

$$D^n(F) \simeq \varprojlim_m \Sigma^{mn}(\mathrm{cr}^n(F))(\Omega^m X, \dots, \Omega^m X)$$

*Proof.* Similar to the above argument. □

**Corollary 19.4.10.** *A filtered-colimit preserving functor $F : \mathcal{C} \to \mathcal{D}$ where $\mathcal{C}$ is either compactly generated by $\mathbb{S}$ or stable and generated by $\Sigma^n\mathbb{S}$ for some object $\mathbb{S}$ is $n$-homogeneous iff it is of the form $F(X) = (C \otimes X^{\otimes n})_{hS_n}$.*

*Proof.* Using the above equivalence, we have $F(X) \simeq \mathrm{cr}_n(F)(X, \dots, X)_{hS_n}$. Therefore

$$F(X) \simeq (\mathrm{cr}_n(F)(\mathbb{S}, \dots, \mathbb{S}) \otimes X^{\otimes n})_{hS_n}$$

since $F$ hence $\mathrm{cr}_n(F)$ commutes with filtered colimits. □

**Lemma 19.4.11.** *For any filtered colimit preserving $n$-homogeneous ($n \geq 1$) functor $F : \mathcal{C} \to \mathcal{D}$, there is some $F' : \mathbf{Sp}(\mathcal{C}) \to \mathbf{Sp}(\mathcal{D})$ such that $F = \Omega^\infty F' \Sigma^\infty$.*

*Proof.* By corollary 19.4.9, the image of a $n$-homogeneous functor lie in the subcategory of infinite loopspaces, and hence $F$ factors through $\Omega^\infty : \mathbf{Sp}(\mathcal{D}) \to \mathcal{D}$. Now since $F$ factors as $\mathcal{C} \to \mathbf{Sp}(\mathcal{D}) \to \mathcal{D}$, it also factors through $\Sigma^\infty : \mathcal{C} \to \mathbf{Sp}(\mathcal{C})$, and it suffices to prove that the middle functor remains $n$-homogeneous. This is direct by verifying the cross effects vanishes. □

**Theorem 19.4.12** (Telescopic Tate vanishing)**.** *For $G$ a finite group, and an object $X$ of $\mathbf{Sp}_{T(n)}$ equipped with a $G$-action, then we have a norm map between the homotopy orbits and the homotopy fixed points $\mathrm{Nm} : X_{hG} \to X^{hG}$, and the cofiber of the norm map $X^{tG} = \mathrm{cof}(\mathrm{Nm})$ vanishes.*

*Proof.* Reference. □

**Construction 19.4.13.** Consider stable categories $\mathcal{C}$ and $\mathcal{D}$, a reduced $(n-1)$-excisive functor $E$, an $n$-homogeneous functor $K$, and an extension of $K$ into $E$ (that is, a fiber sequence $K \to F \to E$, which is also equivalent to the data $E \to \Sigma K$).

By theorem 19.4.8, we have $\Sigma K \simeq \Delta_{S_n} \mathrm{cr}_n(\Sigma K)$. Since $X$ is a retract of $X \oplus Y$, we have $F(X)$ a retract of $F(X \oplus Y)$, hence since $\mathcal{D}$ is also stable we have $F(X \oplus Y) \simeq F(X) \oplus K$. Thus $\mathrm{cof}(F(X) \to F(X \oplus Y)) \simeq \mathrm{fib}(F(X \oplus Y) \to F(X))$. Thus by the direct definition of cross effects and the iterative algorithm for total (co)fibers we have the cross effect of $\Sigma K$ and the cocross effect agrees up to suspensions. Thus we have $\Delta^{S_n}(\mathrm{cr}^n(\Sigma K))$ is cohomogeneous, hence every natural transformation from an $(n-1)$-excisive functor $E$ to $\Delta^{S_n}(\Sigma K)$ vanishes. Finally by the long cofiber sequence

$$(\mathrm{cr}_n(K)\Delta)^{tS_n} \to \Delta_{S_n} \mathrm{cr}_n(\Sigma K) \xrightarrow{\mathrm{Nm}} \Delta^{S_n} \mathrm{cr}_n(\Sigma K) \to (\mathrm{cr}_n(\Sigma K)\Delta)^{tS_n}$$

and that the third functor in the cofiber sequence is $n$-cohomogeneous, we have that a map $E \to \Sigma K \simeq \Delta_{S_n}(\mathrm{cr}_n(\Sigma K))$ is equivalent to a map $E \to (\mathrm{cr}_n(K)\Delta)^{tS_n}$.

As a conclusion, a reduced $n$-excisive functor is equivalent to the data of a reduced $(n-1)$-excisive functor $E$, an $n$-homogeneous functor $K$ regarded as an $n$-symmetric functor, and a natural transformation $E \to (\mathrm{cr}_n(K)\Delta)^{tS_n}$. Moreover, this functor can be reconstructed as the fiber

$$\mathrm{fib}(E \to (\mathrm{cr}_n(K)\Delta)^{tS_n} \to \Delta_{S_n} \mathrm{cr}_n(\Sigma K) \simeq \Sigma K)$$

**Corollary 19.4.14.** *Every $k$-excisive functor $F : \mathbf{Sp}_{T(n)} \to \mathbf{Sp}_{T(n)}$ splits as the direct sum of $j$-homogeneous functors (where $j < k$).*

*Proof.* Follows directly form theorem 19.4.12 and construction 19.4.13. □

## 19.5 Dual calculus in the $T(n)$-local category and nilpotence

corollary 19.4.14 in the previous section implies that the Goodwillie tower of an $n$-excisive functor of $T(n)$-local spectra splits, which implies that the Goodwillie tower of a coanalytic functor splits. In this section, we wish to see that the converse result, that is, every split Goodwillie tower result in a coanalytic functor, then clearly it suffices to prove the following result.

**Theorem 19.5.1** (Mathew). *If $F \simeq \bigoplus F_j$, and $F_j : \mathbf{Sp}_{T(n)} \to \mathbf{Sp}_{T(n)}$ is filtered colimit preserving and $j$-homogeneous, then the natural map*

$$P^k(\bigoplus_{j=1}^{\infty} F_j) \simeq P^k F$$

*is an equivalence. Dually we have*

$$P_k(\prod_{j=1}^{\infty} F_j) \simeq \prod_{j=1}^{k} F_j$$

*(though this dual is not needed later, we still present it here for the sake of symmetry).*

We first present the core lemma of uniform nilpotence used to prove the theorem.

**Lemma 19.5.2.** *Let $F$ be a finite type $n$-spectrum. Then there exists a constant $C$ such that for all $j > k$ and any filtered colimit preserving $j$-homogeneous functor $H : \mathbf{Sp}_{T(n)} \to \mathbf{Sp}_{T(n)}$, we have the natural transformation*

$$F \otimes (\Sigma^{Ck} \operatorname{cr}^k_{\Delta}(H)\Omega^C) \to F \otimes (\operatorname{cr}^k_{\Delta}(H))$$

*is null. This constant $C$ only depends on $F$.*

*Proof.* We first show that lemma 19.5.2 implies theorem 19.5.1. We first note that theorem 19.5.1 is equivalent to $D^k(\bigoplus_{j>k} F_j) \simeq 0$ for all $k$. Then we see that it suffices to prove that $D^k$ commutes with infinite direct sums, which is a property that ordinary sequential limits do not possess (for the construction of $D^k$ as a sequential limit, see corollary 19.4.9). But if we have lemma 19.5.2, this would give the fact that the inverse system $V \otimes \bigoplus(\Sigma^{Ck} \operatorname{cr}^k(F_j)\Delta\Omega^C)$ is nilpotent. Furthermore, since $V$ is a finite spectrum, we have $V \otimes -$ commutes with sequential limits. Therefore

$$0 \simeq \varprojlim V \otimes \bigoplus(\Sigma^{nk} \operatorname{cr}^k(F_j)\Delta\Omega^n) \simeq V \otimes \varprojlim \bigoplus(\Sigma^{nk} \operatorname{cr}^k(F_j)\Delta\Omega^n)$$

Then by taking the telescopic localization of $V$, we have $\varprojlim \bigoplus(\Sigma^{nk} \operatorname{cr}^k(F_j)\Delta\Omega^n)$ is $T(n)$-equivalent to 0, which proves theorem 19.5.1. □

Now it suffices to prove the lemma. To prove this, we need the language of nilpotence and apply it to naive equivariant homotopy theory.

**Definition 19.5.3.** We define a thick tensor ideal $\mathcal{T}^{\otimes}$ of $\mathcal{C}$ to be a thick subcategory which is also closed under tensoring with any object in $\mathcal{C}$, that is, for $X \in \mathcal{T}^{\otimes}$ and $Y \in \mathcal{C}$, $X \otimes Y$ is also in $\mathcal{T}^{\otimes}$.

**Construction 19.5.4.** For a symmetric monoidal category $\mathcal{C}$, recall the definition and the process of generation of thick subcategories in definition 11.4.1 and construction 11.4.2.

For a collection of objects $\mathcal{F} \subseteq \mathcal{C}$, since the intersection of two thick tensor ideals is still a thick tensor ideal, we may again consider the minimal thick tensor ideal containing $\mathcal{F}$, denoted by $\mathcal{T}^{\otimes}(\mathcal{F})$. We will call it the thick tensor ideal generated by $\mathcal{F}$, constructed as follows.

Let $\mathcal{T}^{\otimes}(\mathcal{F}) = \mathcal{T}(\mathcal{F}^{\otimes})$, where $\mathcal{F}^{\otimes}$ is the collections of objects in $\mathcal{C}$ of the form $X \otimes Y$, where $X \in \mathcal{C}$ and $\Sigma^n Y \in \mathcal{F}$ for some $n \in \mathbb{Z}$. We can easily check that this construction satisfies the desired properties. Moreover, the filtration $\mathcal{T}_n(\mathcal{F})$ of $\mathcal{T}(\mathcal{F})$ induces a filtration on $\mathcal{T}^{\otimes}(\mathcal{F})$ by letting $\mathcal{T}_n^{\otimes}(\mathcal{F}) = \mathcal{T}_n(\mathcal{F}^{\otimes})$. By definition, each $\mathcal{T}_n^{\otimes}(\mathcal{F})$ is a tensor ideal of $\mathcal{C}$.

**Definition 19.5.5.** For an object $A \in \mathcal{C}$, we define the subcategory of $A$-nilpotent objects to be $\mathcal{T}^{\otimes}(A)$, or equivalently, $\mathcal{T}(A \otimes \mathcal{C})$. We say it is $A$-nilpotent of degree $n$ if $n$ is the minimal natural number such that it lies in $\mathcal{T}_n^{\otimes}(A)$. Clearly, every nilpotent object is nilpotent of degree $n$ for some $n$.

**Lemma 19.5.6.** *Let $I = \mathrm{fib}(1 \to A)$. Then for an object $X \in \mathcal{C}$, the following are equivalent:*

1. *$X \in \mathcal{T}_n^{\otimes}(A)$.*
2. *$X \otimes I^{\otimes n} \to X$ is null.*
3. *For every sequence of maps $X_n \to X_{n-1} \to \cdots \to X_0 \simeq X$ with $A \otimes X_k \to A \otimes X_{k-1}$ null, the composition $X_n \to \cdots \to X$ is null.*

*Proof.* (1) $\implies$ (2): For $n = 0$, the assertion is trivial. If $n = 1$, then we have the following retract diagram

$$X \xrightarrow{f} A \otimes Y \xrightarrow{g} X$$

Thus since $X \otimes I \to X \to X \otimes A$ is a fiber sequence, it suffices to prove $X \to X \otimes A$ admit a section, and hence $X \otimes A \simeq X \oplus \Sigma(X \otimes I)$, which implies $X \otimes I \to X$ is trivial. For this property, note

$$X \xrightarrow{\eta \otimes \mathrm{id}} A \otimes X \xrightarrow{\mathrm{id} \otimes f} A \otimes A \otimes Y \xrightarrow{\mu \otimes 1} A \otimes Y$$

is equivalent to $f$ by definition, since $\eta : 1 \to A$ is the unit map and $\mu : A \otimes A \to A$ is the multiplication map. Therefore,

$$X \xrightarrow{\eta \otimes \mathrm{id}} A \otimes X \xrightarrow{\mathrm{id} \otimes f} A \otimes A \otimes Y \xrightarrow{\mu \otimes 1} A \otimes Y \xrightarrow{g} X$$

exhibits $X$ as a retract of $A \otimes X$. For the general case, we apply induction and suppose that the case $n$ is solved. Then for the case of $n+1$, suppose $X$ is $A$-nilpotent of exponent $n+1$, let $Y \to X \to Z$ be a fiber sequence where $Y$ is $A$-nilpotent of exponent 1 and $Z$ is $A$-nilpotent of exponent $n$. Then we have the following diagram where the columns are fiber sequences

$$\begin{tikzcd}
Y \otimes I^{\otimes n+1} \arrow[r] \arrow[d] & Y \otimes I \arrow[r, "0"] \arrow[d] & Y \arrow[d] \\
X \otimes I^{\otimes n+1} \arrow[r] \arrow[d] & X \otimes I \arrow[r] \arrow[d] & X \arrow[d] \\
Z \otimes I^{\otimes n+1} \arrow[r, "0"'] & Z \otimes I \arrow[r] & Z
\end{tikzcd}$$

Then note that the map $X \otimes I^{\otimes n+1} \to Z \otimes I$ is null, hence lifts to a map $X \otimes I^{\otimes n+1} \to Y \otimes I$. Therefore, the map $X \otimes I^{\otimes n+1} \to X$ factors as

$$X \otimes I^{\otimes n+1} \to Y \otimes I \xrightarrow{0} Y \to X$$

hence is null. The case where $X'$ is the retract of the above $X$ is direct.

(2) $\implies$ (1): Suppose $\psi : X \otimes I^{\otimes n} \to X$ is null. Then $X$ is a retract of $\mathrm{cof}\,\psi$, hence it suffices to prove $\mathrm{cof}\,\psi \in \mathcal{T}_n^{\otimes}(A)$. To prove this, it suffices to prove $\mathrm{cof}(I^{\otimes n} \to 1) \in \mathcal{T}_n^{\otimes}(A)$, which is the extension of

$$\mathrm{cof}(I^{\otimes k+1} \to I^{\otimes k}) \simeq A \otimes I^{\otimes k} \in \mathcal{T}_1^{\otimes}(A)$$

(2) $\Longrightarrow$ (3): consider the following diagram

$$\begin{tikzcd}
 & X \otimes I \arrow[d] \\
X_1 \arrow[r] \arrow[d] \arrow[ur, dashed] & X \arrow[d] \\
X_1 \otimes A \arrow[r] & X \otimes A
\end{tikzcd}$$

since the bottom map is null, the map $X_1 \to X \to X \otimes A$ is null, hence lifts to $X_1 \to X \otimes I$. Likewise, we may lift the composition $X_2 \to X_1 \to X \otimes I$ to $X_2 \to X \otimes I^{\otimes 2}$ by the same process. Inductively, we result in the fact that the map $X_n \to X$ factors as $X_n \to X \otimes I^{\otimes n} \to X$, hence is null by assumption.

(3) $\Longrightarrow$ (2): Note that the following sequence

$$X \otimes I^{\otimes n} \to X \otimes I^{\otimes n-1} \to \cdots \to X$$

satisfy the desired assumption. □

Then to apply the above definitions to equivariant settings, we apply another version of the monadicity theorem.

**Theorem 19.5.7.** *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjunction of idempotent complete symmetric monoidal stable categories, and $F$ is symmetric monoidal (this implies that $G$ is lax symmetric monoidal). Then if the counit $\varepsilon : FG \to \mathrm{id}_{\mathcal{D}}$ admit a section $\xi : \mathrm{id}_{\mathcal{D}} \to FG$, and that the projection formula $G(X) \otimes Y \simeq G(X \otimes F(Y))$ holds, then the adjunction is monadic, and the monad can be identified with the monad $G(1) \otimes -$ (here $1$ is the tensor unit of $\mathcal{D}$, and $G(1)$ admits a natural algebra structure since $G$ is lax symmetric monoidal).*

*Proof.* We first notice that the section of $\varepsilon$ gives a section of the monad $T = GF$, explicitly $G\xi F : T \to TT$ satisfies

$$T \xrightarrow{G\xi F} TT \xrightarrow{\mu} T$$

is the identity.

Then for any $T$-module $X$, note that the structure map $m : TX \to X$ admit a natural section $X \to TX$ given by the composition

$$X \xrightarrow{\eta_X} TX \xrightarrow{(G\xi F)_X} T^2X \xrightarrow{Tm_X} TX$$

This is a section of $m : TX \to X$ by definition of $G\xi F$. Moreover this natural section is $T$-linear, hence we have $X$ a retract of $TX$ for every $T$-module $X$. Therefore the inclusion $\mathbf{LMod}_T^{\mathrm{free}}(\mathcal{C}) \hookrightarrow \mathbf{LMod}_T(\mathcal{C})$ is an equivalence after applying idempotent completion, since it is fully faithful (by definition) and every object of $\mathbf{LMod}_T(\mathcal{C})$ is a retract of an object of $\mathbf{LMod}_T^{\mathrm{free}}(\mathcal{C})$.

We then prove that the map $K : \mathbf{LMod}_T^{\mathrm{free}}(\mathcal{C}) \to \mathcal{D}$ (recall its construction in construction 19.3.2) is an equivalence after applying idempotent completion. Firstly we have seen in construction 19.3.2 that $K$ is fully faithful. Then for an object of $\mathcal{D}$, the section $\xi$ provides

$$D \xrightarrow{\xi_D} FGD \xrightarrow{\varepsilon_D} D$$

hence we have $D$ a retract of $FGD \simeq KGD$.

Hence we have $E : \mathcal{D} \to \mathbf{LMod}_T(\mathcal{C})$ an equivalence after applying idempotent completion. But by assumption both sides are already idempotent complete, so the adjunction is monadic.

It then remains to identify the monads. Namely, by the projection formula, we have

$$G(1) \otimes - \simeq G(1 \otimes F-) \simeq GF-$$

the identification of the units and multiplications follows by diagram chasing. □

**Lemma 19.5.8.** *Let $f : R \to S$ be a morphism of $\mathbb{E}_1$-rings. Suppose that there is a map of $R - R$-bimodules $g : S \to R$ such that $gf = 1$, then the forgetful functor is left adjoint to* $\mathrm{Hom}_R(S, -) : \mathbf{LMod}_R \to \mathbf{LMod}_S$ *($\mathrm{Hom}_R(S, M)$ is equipped with the left $S$-module structure via right action on the first factor), and the counit admits a natural section.*

*Proof.* Firstly the adjunction is obvious. Then the counit of the adjunction is given by

$$f^* : \mathrm{Hom}_R(S, -) \to \mathrm{Hom}_R(R, -)$$

Then we claim that $g^*$ is the section as desired. Firstly, note that

$$g^* : \mathrm{Hom}_R(R, -) \to \mathrm{Hom}_R(S, -)$$

is a morphism of left $R$-modules since $g : S \to R$ is a morphism of $R - R$-bimodules, and in particular preserves the right $R$-module structure. Then $f^* g^* = (gf)^* = \mathrm{id}$, so the counit admits a natural section. □

**Theorem 19.5.9.** *For $G$ a finite group, let $H$ be a subgroup of $G$ (this is in fact true for stably dualizable group and a finite index subgroup), then we have $\Sigma^\infty_+ G/H$ is a natural cocommutative coalgebra given by the diagonal map, and hence a natural commutative algebra structure on $\mathbb{D}\,\Sigma^\infty_+ G/H$. Then we have*

$$\mathbf{Sp}^{BH} \simeq \mathbf{Mod}_{\mathbb{D}\,\Sigma^\infty_+ G/H}(\mathbf{Sp}^{BG})$$

*where $\mathbf{Sp}^{BG}$ is the category of naive $G$-spectra.*

*Remark* 29. This argument also applies to the category of genuine $G$-spectra with some care.

*Proof.* We here regard $\mathbf{Sp}^{BG}$ as the category $\mathbf{LMod}_{\mathbb{S}[G]}(\mathbf{Sp})$ (here $\mathbb{S}[G] := \Sigma^\infty_+ G$ is the $\mathbb{E}_1$-group algebra), equipped wit. Then note that the restriction functor is given by the forgetful functor of modules induced by the natural map $\mathbb{S}[H] \hookrightarrow \mathbb{S}[G]$. The coinduction is then given by

$$\mathrm{coind}^G_H(M) \simeq \mathrm{Hom}_{\mathbb{S}[H]}(\mathbb{S}[G], M)$$

By theorem 19.5.7, it suffices to show that the counit $\varepsilon : \mathrm{res}^G_H \circ \mathrm{coind}^G_H \to \mathrm{id}$ admits a section (the projection formula holds by formal reasons). But in fact, the section comes from lemma 19.5.8, since the inclusion $\mathbb{S}[H] \hookrightarrow \mathbb{S}[G]$ has obviously a $\mathbb{S}[H] - \mathbb{S}[H]$-bimodule which is a left inverse given by

$$\mathbb{S}[G] \simeq \bigoplus_{g \in G} \mathbb{S} \to \bigoplus_{h \in H} \mathbb{S} \simeq \mathbb{S}[H]$$

which is the identity on $\mathbb{S}$ indexed by $h \in H$ and null on $\mathbb{S}$ indexed by $g \notin H$. Therefore by theorem 19.5.7 this adjunction is monadic, and it is direct to verify the monad as the algebra $\Sigma^\infty_+ G/H$. □

**Construction 19.5.10.** We apply theorem 19.5.9 to nilpotence. Let $\mathcal{F}$ be a collection of subgroups of $G$. We say a object of $\mathbf{Sp}^{BG}$ is $\mathcal{F}$-nilpotent of degree $n$ if it is $\prod_{H \in \mathcal{F}} (\Sigma^\infty_+ G/H)$-nilpotent of degree $n$. We will mainly apply the case where $\mathcal{F}$ consists of a single subgroup $H \leq G$.

**Theorem 19.5.11.** *Any finite spectrum $F$ in $\mathbf{Sp}_{T(n)}$ equipped with the trivial $G$-action is nilpotent viewed as an element of $\mathbf{Sp}^{BG}$.*

*Proof.* We consider the simplicial model for a point with free $G$-action, namely, let $EG_k = G^{k+1}$, and $EG = \varinjlim EG_{\leq k}$. Then since we are working in the $T(n)$-local category, by theorem 19.4.12 we have

$$\mathbb{S}^{hG}_{T(n)} \simeq (\mathbb{S}_{T(n)})_{hG} \simeq (\varinjlim (\mathbb{S}_{T(n)} \otimes EG_{\leq k}))_{hG} \simeq \varinjlim (\mathbb{S}_{T(n)} \otimes EG_{\leq k})_{hG} \simeq \varinjlim (\mathbb{S}_{T(n)} \otimes EG_{\leq k})^{hG}$$

By compactness of $F$, the map $F \hookrightarrow F \otimes \mathbb{S}_{T(n)}^{hG}$ factors as

$$F \to F \otimes (\mathbb{S}_{T(n)} \otimes EG_{\leq k})^{hG} \to F \otimes \mathbb{S}_{T(n)}^{hG}$$

Since $F$ is finite, and homotopy fixed points agrees with the homotopy orbits, hence $F \otimes \mathbb{S}_{T(n)}^{hG} \simeq (F \otimes \mathbb{S}_{T(n)})^{hG}$, and the middle term satisfy a similar reduction. By the adjunction $\Delta \dashv \varprojlim$ (here interpreted as the giving a spectrum the trivial $G$-action is left adjoint to taking homotopy fixed points), we obtain a retract diagram

$$F \to F \otimes EG_{\leq k} \to F$$

but $F \otimes EG_{\leq k}$ is clearly $G$-nilpotent by its construction, which implies that so is $F$. $\square$

**Lemma 19.5.12.** *For a finite spectrum $F$ (equipped with the trivial action), there is some constant $C$ only depending on $F$ such that for any finite group and a subgroup $H \leq G$ with $v_p(G) - v_p(H) \leq 1$ (here $v_p(G)$ is the $p$-adic valuation of the order of $G$), and any object $X \in \mathbf{Sp}_{T(n)}^{BG}$, we have $\exp_H(F \otimes X) \leq C$.*

*Proof.* Consider the Sylow-$p$-subgroup $G_p \leq G$ and $H_p \leq H$, and we assume $H_p \leq G_p$ (which would imply $H_p \triangleleft G_p$ by some finite group theory). Consider $\operatorname{coind}_{G_p}^{G}(\mathbb{D}\,\Sigma_+^\infty(G_p/H_p)) = \mathbb{D}\,\Sigma_+^\infty(G/H_p)$. Note that $G/H_p$ projects naturally to $G/H$, and by sending $gH$ to the sum of $ghH_p$ (where $hH_p$ runs over the cosets of $H_p$ in $H$), we obtain that $\mathbb{D}\,\Sigma_+^\infty(G/H)$ is a retract of $\mathbb{D}\,\Sigma_+^\infty(G/H_p)$. This implies that $\exp_{H_p}(\operatorname{res}_{G_p}^{G} X \otimes F) \geq \exp_H(X \otimes F)$, since if $\operatorname{res}_{G_p}^{G} X \otimes F \in \mathcal{T}_n^{\otimes}(\Sigma_+^\infty(G/H_p))$, by the same generation process, after the retraction, we would have $X \otimes F \in \mathcal{T}_n^{\otimes}(\Sigma_+^\infty(G/H))$.

Therefore we may pass to the case where $G$ is a $p$-group, and hence $H \leq G$ is an index $p$ subgroup of $G$, and hence normal. Therefore $\exp_H(F \otimes X) \leq \exp_H(F)$, and $\exp_H(F) \leq \exp(F)$, where the right hand side is the exponent of $F$ as a $C_p$-spectra. Therefore taking $C = \exp(F)$ gives the desired result. $\square$

Now we are ready to prove lemma 19.5.2

*Proof.* Firstly let $\mathbb{S}_{T(n)}^{\rho_j}$ denote the representation sphere of the standard representation of the symmetry group $S_j$ (that is, $\Sigma_j \curvearrowright \mathbb{R}^{j-1}$). Then note that by the universal property of loopspaces, we have a $S_j$-equivariant map

$$e : \mathbb{S}_{T(n)} \simeq \mathbb{S}_{T(n)}^{\otimes j} \to \Omega(\Sigma \mathbb{S}_{T(n)})^{\otimes j} \simeq \mathbb{S}_{T(n)}^{\rho_j}$$

This is, in fact, the Euler class for this representation of $S_j$, which vanishes if the representation contains a trivial subrepresentation. We claim that $F \otimes e^C : F \to F \otimes \mathbb{S}_{T(n)}^{C\rho_j}$ is null-homotopic. This is since we may choose a Young subgroup $H \leq S_j$ with $v_p(S_j) - v_p(H) \leq 1$, and the restriction of the standard representation to any Young subgroup of $S_j$ admits a trivial subrepresentation. Therefore $e$ is null if restricted to $\mathbf{Sp}^{BH}$. By lemma 19.5.6, we have $V \otimes e^C$ null in $\mathbf{Sp}^{BS_j}$ for $j \geq 2$ (note $j \geq 2$ is essential since otherwise the representation would be trivial anyway, and the Euler class would be the identity).

Then, we show that the natural transformation

$$F \otimes (\Sigma^{Ck} \operatorname{cr}_\Delta^k(H)\Omega^C) \to F \otimes (\operatorname{cr}_\Delta^k(H))$$

is null. By the equivalence $H \simeq (L \circ \Delta)_{hS_j}$ as in theorem 19.4.8 (where $L = \operatorname{cr}^j(H)$), it suffices to prove that the $S_j$-equivariant natural transformation

$$F \otimes (\Sigma^{Ck} \operatorname{cr}_\Delta^k(L \circ \Delta_j \circ \Omega^C)) \to F \otimes \operatorname{cr}_\Delta^k(L \circ \Delta_j)$$

is null (the original case follows from taking $S_j$-homotopy orbits of this one). Since $L$ preserves filtered colimits in each variables by assumption, we have $L(X_1, \dots, X_j) \simeq \partial L \otimes X_1 \otimes \cdots \otimes X_j$. By directly checking the definition of cocross effects, we have

$$\Sigma^{Ck} \operatorname{cr}^k_\Delta(L \circ \Delta_j \circ \Omega^C) \simeq \Big( \bigoplus_{f:[j]\twoheadrightarrow[k]} \mathbb{S}^{-C\rho_{f^{-1}(1)}}_{T(n)} \otimes \cdots \otimes \mathbb{S}^{-C\rho_{f^{-1}(k)}}_{T(n)} \Big) \otimes (L \circ \Delta_j)$$

as $S_j$-equivariant objects (where the $S_j$ action is given by the permutation on $[j] = \{1, 2, \dots, j\}$, and in case that some $\sigma \in S_j$ preserves some surjection $f : [j] \twoheadrightarrow [k]$, it must lie in the subgroup $\Sigma_{f^{-1}(1)} \times \cdots \times \Sigma_{f^{-1}(k)}$, and thus acts on the permutation spheres). Similarly,

$$\operatorname{cr}^k_\Delta(L \circ \Delta_j) \simeq \Big( \bigoplus_{f:[j]\twoheadrightarrow[k]} \mathbb{S}^{\otimes f^{-1}(1)}_{T(n)} \otimes \cdots \otimes \mathbb{S}^{\otimes f^{-1}(k)}_{T(n)} \Big) \otimes (L \circ \Delta_j)$$

where the action on $\mathbb{S}_{T(n)}$ is given likewise. Thus to show the desired equivariant map vanishes, it suffices to prove that for each Young subgroup $\Sigma_{j_1} \times \cdots \times \Sigma_{j_k}$, we have the equivariant map

$$F \otimes \mathbb{S}^{-C\rho_{j_1}}_{T(n)} \otimes \cdots \otimes \mathbb{S}^{-C\rho_{j_k}}_{T(n)} \to F \otimes \mathbb{S}_{T(n)}$$

vanishes. But taking dual of the above, we obtain

$$\mathbb{D}\, F \otimes e^C_1 \otimes \cdots \otimes e^C_k : \mathbb{D}\, F \to \mathbb{D}\, F \otimes \mathbb{S}^{C\rho_{j_1}}_{T(n)} \otimes \cdots \otimes \mathbb{S}^{C\rho_{j_k}}_{T(n)}$$

Now by the assumption $j > k$, we have at least one $j_i \geq 2$, hence $\mathbb{D}\, F \otimes \mathbb{S}_{T(n)} \to \mathbb{D}\, F \otimes \mathbb{S}^{C\rho_{j_i}}_{T(n)}$ vanishes, hence the above map vanishes if we let $C$ be the constant as in lemma 19.5.12 for $\mathbb{D}\, F$. □

**Corollary 19.5.13.** *Let $F : \mathbf{Sp}_{T(n)} \to \mathbf{Sp}_{T(n)}$ be a reduced functor preserving filtered colimits. Then $F$ is coanalytic (i.e. $\varinjlim P^k F \simeq F$) iff $F$ is of the form*

$$F(X) = \bigoplus (C_j \otimes X^{\otimes j})_{hS_j}$$

*where $C_j$ is an $S_j$-equivariant $T(n)$-local spectra, and the $S_j$-action on $X^{\otimes j}$ is given by permutation of the factors.*

*Proof.* $\Longrightarrow$ :

$$F(X) \simeq \varinjlim_k P^k F(X) \simeq \varinjlim_k \bigoplus_{j<k} D_j P^k F(X) \simeq \varinjlim_k \bigoplus_j j < k(C(j) \otimes X^{\otimes j}) \simeq \bigoplus_j (C(j) \otimes X^{\otimes j})$$

$\Longleftarrow$ : By theorem 19.5.1,

$$D^j(\bigoplus) \simeq D^j(\bigoplus F_k) \simeq F_j$$

□

## 19.6 Classical operads and deduction of the final statement

**Definition 19.6.1.** We define a classical operad of a category $\mathcal{C}$ to be an $\mathbb{E}_1$-algebra in the category of symmetric sequences in $\mathcal{C}$, that is, $\mathrm{Fun}(\mathrm{Core}\,\mathbf{Fin}, \mathcal{C})$, with respect to the composition product. A classical cooperad is a $\mathbb{E}_1$-coalgebra with respect to the composition product.

**Construction 19.6.2.** Note that the functor $\mathrm{Fun}(\mathrm{Core}\,\mathbf{Fin}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{C})$ given by

$$F \mapsto (X \mapsto \bigoplus (F(n) \otimes X^{\otimes n})_{hS_n})$$

is symmetric monoidal, so any classical operad gives rise to a monad in $\mathcal{C}$. corollary 19.5.13 shows that when $\mathcal{C} = \mathbf{Sp}_{T(n)}$, this inclusion is fully faithful with image the coanalytic functors, and $P^\infty = \varinjlim P^n$ is the associated coreflection functor (i.e. the right localization onto the coanalytic functors).

**Construction 19.6.3.** Recall that for any augmented $\mathbb{E}_1$-algebra of $\mathcal{C}$, say $f : A \to 1$, we define the bar construction of $A$ to be the realization of the simplicial object

$$\mathrm{Bar}(A) = \quad \cdots \xrightarrow[\cdots]{\cdots} A \otimes A \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} A \rightrightarrows 1$$

By definition this is the relative tensor $1 \otimes_A 1$, and it admits a natural augmented $\mathbb{E}_1$-coalgebra structure given by

$$1 \otimes_A 1 \simeq 1 \otimes_A A \otimes_A 1 \to 1 \otimes_A 1 \otimes_A 1 \simeq 1 \otimes_A 1 \otimes 1 \otimes_A 1$$

Similarly, for an augmented $\mathbb{E}_1$-coalgebra $B$ of $\mathcal{C}$, we may likewise define its cobar construction

$$\mathrm{coBar}(B) = \quad 1 \rightrightarrows B \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} B \otimes B \xrightarrow[\cdots]{\cdots} \cdots$$

This is likewise an augmented $\mathbb{E}_1$-algebra $\mathrm{coBar}(B)$ of $\mathcal{C}$. By definition, we have $\mathrm{Bar} : \mathrm{Alg}(\mathcal{C}) \rightleftarrows \mathrm{coAlg}(\mathcal{C}) : \mathrm{coBar}$.

**Theorem 19.6.4.** *The functor $L_{T(n)}\Sigma^\infty_{T(n)}\Omega^\infty_{T(n)} : \mathbf{Sp}_{T(n)} \to \mathbf{Sp}_{T(n)}$ (recall this construction in corollary 19.1.24) is coanalytic. More precisely, we have*

$$L_{T(n)}\Sigma^\infty_{T(n)}\Omega^\infty_{T(n)}X \simeq L_{T(n)} \bigoplus_{k=1}^{\infty} (X^{\otimes}k)_{hS_k}$$

*Proof.* For a $T(n)$-local spectrum $Y$, we apply $\Phi$ (the Bousfield-Kuhn functor) to the unit map $\Omega^\infty Y \to \Omega^\infty\Sigma^\infty\Omega^\infty Y$, and since $\Phi\Omega \simeq L_{T(n)}$ by theorem 19.2.6, we have a natural transformation $\lambda_Y : Y \to L_{T(n)}\Sigma^\infty\Omega^\infty Y$. Note that $L_{T(n)}\Sigma^\infty\Omega^\infty Y$ is a non-unital $\mathbb{E}_\infty$-ring spectrum, hence $\lambda$ extends to a map

$$L_{T(n)} \bigoplus_{k=1}^{\infty} (Y^{\otimes k}) \to L_{T(n)}\Sigma^\infty\Omega^\infty Y$$

Now, by a result of Kuhn this is an equivalence when $X$ is $d_{n+1}$-connected (where $d_{n+1}$ is some constant depending on the inclusion of $\mathbf{An}_*^{v_n} \hookrightarrow \mathbf{An}_*$, or the connectivity of some chosen finite space of type $n+1$) and $T(i)_*X = 0$ for $0 < i < n$. By corollary 19.1.24 and picking the corresponding inclusion of $\mathbf{An}_*^{v_n}$ into $\mathbf{An}_*$ we have alternatively $\Omega^\infty_{T(n)}$ equivalent to $\tau_{\geq d_{n+1}+1}\Omega^\infty : L_{T(n)}\mathbf{Sp} \to \mathbf{An}_*^{v_n}$, hence we have

$$\Sigma^\infty_{T(n)}\Omega^\infty_{T(n)}X \simeq L_{T(n)}\Sigma^\infty\Omega^\infty(\tau_{\geq d+1}X)$$

since $\tau_{\geq d+1}X$ is $T(n)$-local, it is $T(i)$-acyclic for $i < n$, and hence we have

$$L_{T(n)}\Sigma^\infty_{T(n)}\Omega^\infty_{T(n)}(X) \simeq L_{T(n)} \bigoplus_{k=1}^{\infty} (\tau_{\geq d_{n+1}+1}(X)^{\otimes k})_{hS_k}$$

Via the equivalence $L_{T(n)}\tau_{\geq d_{n+1}}X \simeq L_{T(n)}X$, the conclusion holds. $\square$

**Definition 19.6.5.** We define the cocommutative cooperad to be $\Sigma^\infty_{T(n)}\Omega^\infty_{T(n)}$ (this is coanalytic, and is also a comonad since it is the composition of a right adjoint by its left adjoint). We define the Lie operad to be its cobar construction.

*Remark* 30. Algebras of Lie operads behaves like Lie algebras, and hence the final theorem would give a categorical characterization of Whitehead products.

**Theorem 19.6.6.** *Let $F : \mathbf{Sp}_{T(n)} \to \mathbf{Sp}_{T(n)}$ be a functor that preserves sifted colimits. Then $F$ is coanalytic.*

*Proof.* We prove $F$ is a colimit of coanalytic functors, which is coanalytic since the inclusion of coanalytic functors into all functors preserves colimits. Note that for any spectrum $X$, the natural map

$$\varinjlim_k \Sigma^{\infty-k}\Omega^{\infty-k}(\tau_{\geq d_{n+1}+1}X) \to X$$

is an equivalence (here $d_{n+1}$ is defined as in the above proof, and is related to the explicit inclusion $\mathbf{An}_*^{v_n} \hookrightarrow \mathbf{An}_*$. ), since this obviously holds for the case of a finite spectrum, and the general case follows from writing $X$ a filtered colimit of finite spectra. Therefore, it suffices to prove that the functor

$$X \mapsto F(L_{T(n)}\Sigma^{\infty-k}\Omega^{\infty-k}(\tau_{\geq d_{n+1}+1}X))$$

is coanalytic.

Since $F$ preserves sifted colimits, and $L_{T(n)}$ and $\Sigma^{\infty-k}$ are left adjoints, we have $F \circ L_{T(n)} \circ \Sigma^{\infty-k} : \mathbf{An}_* \to \mathbf{Sp}_{T(n)}$ preserves sifted colimits. Since $\mathbf{An}_*$ is projectively generated with compact generators given by $\mathbf{Fin}_*$, $F \circ L_{T(n)} \circ \Sigma^{\infty-k}$ is the left Kan extension along the inclusion $i : \mathbf{Fin}_* \hookrightarrow \mathbf{An}_*$ of its restriction $\mathbf{Fin}_* \to \mathbf{Sp}_{T(n)}$, which we denote by $f$. Now, recall that to calculate a left Kan extension, we may apply the coend formula, which is given as follows

$$\mathrm{Lan}_i f(X) \simeq \int^{I \in \mathbf{Fin}_*} \mathrm{Hom}_*(I, X) \otimes f(I)$$

here this tensor is the tensor of a space with a spectrum. Recall also that we may reduce a coend to a colimit by passing to the twisted arrow category $T(\mathbf{Fin}_*)$ (that is, the straightening of $\mathrm{Hom} : \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathbf{An}$), explicitly given by

$$T(\mathbf{Fin}_*) \to \mathbf{Fin}_*^{\mathrm{op}} \times \mathbf{Fin}_* \xrightarrow{\mathrm{Hom}(-,X)\times f} \mathbf{An}_* \times \mathbf{Sp}_{T(n)} \xrightarrow{\otimes} \mathbf{Sp}_{T(n)}$$

Thus, $F \circ L_{T(n)} \circ \Sigma^{\infty-k}$ can be written as the colimit of functors of the form

$$\mathrm{Hom}_*(I, -) \otimes F(L_{T(n)}\Sigma^{\infty-k}J) : \mathbf{An}_* \to \mathbf{Sp}_{T(n)}$$

where $I$ and $J$ are finite pointed sets. This implies that

$$X \mapsto F(L_{T(n)}\Sigma^{\infty-k}\Omega^{\infty-k}(\tau_{\geq d_{n+1}+1}X))$$

can be written as the colimit of functors of the form

$$X \mapsto \mathrm{Hom}_*(I, \Omega^{\infty-k}(\tau_{\geq d_{n+1}+1}X)) \otimes C$$

for $C$ some $T(n)$-local spectrum. It suffices to show that

$$X \mapsto L_{T(n)}\Sigma^\infty \mathrm{Hom}_*(I, \Omega^{\infty-k}(\tau_{\geq d_{n+1}+1}X))$$

are coanalytic. However, since $I$ is a finite set, the right hand side can be identified with $(\Sigma^\infty\Omega^{\infty-k}(\tau_{\geq d_{n+1}+1}X))^{\otimes I-*}$. But this functor is clearly coanalytic by the same process of proof as in theorem 19.6.4. ☐

**Theorem 19.6.7.** *There is a map of operads $\gamma : \Phi\Theta \to \mathrm{coBar}(\Sigma^\infty_{T(n)}\Omega^\infty_{T(n)})$ which is an equivalence of operads. Therefore, $\Phi\Theta$ can be identified with the Lie operad on $\mathbf{Sp}_{T(n)}$, which produces $\mathbf{An}^{v_n}_*$ as the category of Lie algebras of $\mathbf{Sp}_{T(n)}$.*

*Proof.* Note that the cobar construction $\mathrm{coBar}(\Sigma^\infty_{T(n)}\Omega^\infty_{T(n)})$ is given by the totalization of the cosimplicial object

$$\mathrm{id}_{\mathbf{Sp}_{T(n)}} \rightrightarrows \Sigma^\infty_{T(n)}\Omega^\infty_{T(n)} \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \Sigma^\infty_{T(n)}\Omega^\infty_{T(n)}\Sigma^\infty_{T(n)}\Omega^\infty_{T(n)} \xrightarrow[\cdots]{\cdots} \cdots$$

But since $\Phi\Omega^\infty_{T(n)} \simeq \Sigma^\infty_{T(n)}\Theta \simeq \mathrm{id}_{\mathbf{Sp}_{T(n)}}$, the totalization of the above cosimplicial diagram is equivalent to the following

$$\Phi\Omega^\infty_{T(n)}\Sigma^\infty_{T(n)}\Theta \rightrightarrows \Phi\Omega^\infty_{T(n)}\Sigma^\infty_{T(n)}\Omega^\infty_{T(n)}\Sigma^\infty_{T(n)}\Theta \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots$$

hence $\mathrm{coBar}(\Sigma^\infty_{T(n)}\Omega^\infty_{T(n)}) \simeq \mathrm{Tot}(\Phi(\Omega^\infty_{T(n)}\Sigma^\infty_{T(n)})^{\bullet+1}\Theta)$, which produces a map

$$\gamma : \Phi\Theta \to \mathrm{Tot}(\Phi(\Omega^\infty_{T(n)}\Sigma^\infty_{T(n)})^{\bullet+1}\Theta)$$

Now upon applying $P_k$ on both sides, since $\Phi$ and $\Theta$ preserves filtered colimits, by the construction of $P_k$, we have $P_k(\Phi\Theta) \simeq \Phi P_k(\mathrm{id}_{\mathbf{An}^{v_n}_*})\Theta$, and on the right hand side we have

$$P_k\,\mathrm{Tot}(\Phi(\Omega^\infty_{T(n)}\Sigma^\infty_{T(n)})^{\bullet+1}\Theta) \simeq \Phi\,\mathrm{Tot}(P_k((\Omega^\infty_{T(n)}\Sigma^\infty_{T(n)})^{\bullet+1}))\Theta$$

Thus it suffices to verify that we have

$$P_k(\mathrm{id}_{\mathbf{An}^{v_n}_*}) \to \mathrm{Tot}(P_k((\Omega^\infty_{T(n)}\Sigma^\infty_{T(n)})^{\bullet+1}))$$

is an equivalence.

In fact, we prove a more general statement saying that

$$P_k(FG) \simeq \mathrm{Tot}(P_k(F(\Omega^\infty_{T(n)}\Sigma^\infty_{T(n)})^{\bullet+1}G))$$

Suppose $F$ is of the form $H\Sigma^\infty_{T(n)}$, then the above is obviously an equivalence by the extra degeneracy argument on cosimplicial objects. Then note that homogeneous functors are of the form $H\Sigma^\infty_{T(n)}$ by lemma 19.4.11. Therefore the assumption holds for all homogeneous functor $F$. Then the conclusion follows from induction and approximating $F$ to the $k$-th degree (note $P_k(P_k(F)) = P_k(F)$), and on each step $D_iF \to P_iF \to P_{i-1}F$ is a fiber sequence, so the homogeneous case and the induction hypothesis works. $\square$

# Part VI

# Ambidexterity

# Chapter 20

# Ambidexterity in terms of $\mathrm{An}_n^m$

## 20.1 Beck-Chevalley fibrations

**Definition 20.1.1.** Recall that a pair of adjunction is simply a bifibration to the category $\Delta^1$. Now, given a functor $q : \mathcal{C} \to \mathcal{X}$ a bifibration, for each $f : X \to Y$ in $\mathcal{X}$, we let $f_! \dashv f^*$ the adjunction associated to $f$ over $f_! : \mathcal{C}_X \rightleftarrows \mathcal{C}_Y : f^*$, where $q^{-1}(X)$ is the homotopy fiber of the map $q$ at $X$. We denote the unit and counit of the adjunction induced by $f$ respectively by $\eta_f$ and $\varepsilon_f$.
Then, for each Cartesian square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

we have a natural equivalence of functors $(g')^* f^* = (f')^* g^*$, which induces a natural transformation by the adjunction

$$f'_!(g')^* \xrightarrow{\eta_f} f'_!(g')^* f^* f_! \simeq f'_!(f')^* g^* f_! \xrightarrow{\varepsilon_{f'}} g^* f_!$$

We denote this natural transformation associated to the Cartesian square $\sigma$ by $BC[\sigma]$, the Beck-Chevalley transformation associated to $\sigma$.
We define $q : \mathcal{C} \to \mathcal{X}$ to be a Beck-Chevalley fibration if it is a bifibration and for all $\sigma$, $BC[\sigma]$ is invertible.

*Remark* 31. The Beck-Chevalley condition can be viewed as a nice behavior with respect to spans. Explicitly, for a bifibration $q : \mathcal{C} \to \mathcal{X}$, and a span in $\mathcal{X}$, say $X \xleftarrow{f} Z \xrightarrow{g} Y$, we can obviously form a functor $g_! f^* : \mathcal{C}_X \to \mathcal{C}_Y$. However, when we compose spans, we would form the pullback

$$\begin{tikzcd} Z \arrow[r, "g_1'"] \arrow[d, "f_2'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y_2 \arrow[r, "g_2"] \arrow[d, "f_2"] & X_3 \\ Y_1 \arrow[r, "g_1"'] \arrow[d, "f_1"'] & X_2 & \\ X_1 & & \end{tikzcd}$$

Then the Beck-Chevalley condition is nothing but stating that the two ways going from $q^{-1}(X_1)$ to $q^{-1}(X_3)$ agrees. We will come back to this in the next section when we are really treating spans.

We will then define the notion of ambidexterity. This is the phenomenon that $f_!$ is not only the left adjoint of $f^*$, but also the right adjoint.

**Construction 20.1.2.** Let $\mathcal{X}$ be a category which admits pullbacks and let $q : \mathcal{C} \to \mathcal{X}$ be a Beck-Chevalley fibration. We will define for $n \geq -2$ two collections of morphisms of $\mathcal{X}$ called the (weakly) $n$-ambidextrous morphisms, and for each $n$-ambidextrous morphism $f : X \to Y$, a natural transformation $\mu_f^{(n)} : \mathrm{id}_{\mathcal{C}_Y} \to f_! \circ f^*$ which exhibits $f_!$ a right adjoint of $f^*$.
For $n = -2$, we define $f$ to be (weakly) $(-2)$-ambidextrous iff $f$ is an equivalence. We define $\mu_f^{(-2)}$ to be the inverse of the transformation $\varepsilon_f : f_! \circ f^* \to \mathrm{id}_{\mathcal{C}_Y}$.
We assume now that we have defined the $n$-ambidextrous morphisms and $\mu_f^{(n)}$. Let $f : X \to Y$ be an arbitrary morphism in $\mathcal{C}$, and let $\delta : X \to X \times_Y X$ be the diagonal, and we have the following diagram

$$\begin{tikzcd}
X \arrow[dr, "\delta"] \arrow[drr, bend left, "\mathrm{id}_X"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\
& X \times_Y X \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Let $\sigma$ denote the square in this diagram. Then since $q$ is a Beck-Chevalley fibration, we have $BC[\sigma] : \pi_{1!}\pi_2^* \to f^* f_!$ admit an inverse $BC[\sigma]^{-1} : f^* f_! \to \pi_{1!}\pi_2^*$. We define $f$ to be weakly $(n+1)$-ambidextrous if $\delta$ is $n$-ambidextrous. In this case, we define a natural transformation

$$\nu_f^{(n+1)} : f^* f_! \xrightarrow{BC[\sigma]^{-1}} \pi_{1!}\pi_2^* \xrightarrow{\mu_\delta^{(n)}} \pi_{1!}\delta_!\delta^*\pi_1^* \simeq \mathrm{id}_{\mathcal{C}_X} \circ \mathrm{id}_{\mathcal{C}_X} \simeq \mathrm{id}_{\mathcal{C}_X}$$

We say $f$ is $(n+1)$-ambidextrous if for every pullback diagram

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

in $\mathcal{X}$, the map $f'$ is weakly $(n+1)$-ambidextrous and $\nu_f^{(n+1)}$ is the counit for the adjunction $(f')^* \dashv f'_!$. In this case, we say $\mu_f^{(n+1)} : \mathrm{id}_{\mathcal{C}_Y} \to f_! f^*$ induced by the adjunction $f^* \dashv f_!$ given by the counit $\nu_f^{(n+1)}$.

The previous construction has many interesting property.

**Proposition 20.1.3.** *1. If $f : X \to Y$ is an $n$-ambidextrous morphism in $\mathcal{X}$, then $f$ is $n$-truncated.*

*2. Let $f : X \to Y$ be a (weakly) $n$-ambidextrous morphism in $\mathcal{X}$, then any pullback of $f$ is also (weakly) $n$-ambidextrous.*

*3. Let $-2 \leq m \leq n$, if $f$ is a (weakly) $m$-ambidextrous morphism in $\mathcal{X}$, then $f$ is (weakly) $n$-ambidextrous. Moreover, the morphisms $\mu_f^{(n)}$ and $\mu_f^{(m)}$ ($\nu_f^{(n)}$ and $\nu_f^{(n)}$) agree.*

*4. Let $-2 \leq m \leq n$, and let $f$ be a (weakly) $n$-ambidextrous morphism in $\mathcal{X}$, then $f$ is (weakly) $m$-ambidextrous iff $f$ is $m$-truncated.*

*Proof.* The first assertion obviously holds for $n = -2$, by definitions. Since $f$ is $n$-ambidextrous implies $\delta$ is $(n-1)$-ambidextrous, hence $\delta$ is $(n-1)$-truncated, hence by HTT 5.5.6.15, we have $f$ is $n$-truncated.
The $n$-ambidextrous part of the second assertion is obvious, and the weakly $n$-ambidextrous part follows from the $n$-ambidextrous part.

For the third assertion, it suffices to prove the case $n = m+1$. Now, the $m$-ambidextrous part follows from weakly $m$-ambidextrous part since it suffices to verify the pullback diagram, which is obvious. Then, notice that the $m$-ambidextrous part implies the weakly $(m+1)$-ambidextrous part, which is by verifying the property on $\delta$. Thus it suffices to prove that for $f$ an equivalence, then $f$ is $(-1)$-ambidextrous. In this case, $f$ is clearly weakly $(-1)$-ambidextrous by the arguments before, and $f_!$ and $f^*$ are homotopy inverses, and a calculation shows that the map $\nu_f$ is the homotopy inverse of the unit map $\eta_f$, thus $f$ is $(-1)$-ambidextrous. Moreover, the maps $\mu_f^{(-1)}$ and $\mu_f^{(-2)}$ agree since they are both inverses to the map $\varepsilon_f : f_! f^* \to \mathcal{C}_Y$.
For the fourth assertion, the weakly ambidextrous part follows from the ambidextrous part. Then the only if direction is by the first assertion. And for the converse, we prove by induction on $m$. We suppose $f$ is $m$-truncated and $n$-ambidextrous for $m \le n$, for $m = -2$, the assertion is obvious. Assume then that $m > -2$, and let $\delta : X \to X \times_Y X$ be the diagonal map. It suffices to prove $\delta$ is $(m-1)$-ambidextrous and that the map $\nu_f^{(m)} : f^* f_! \to \mathrm{id}_{\mathcal{C}_X}$ is the counit for the adjunction. Since $\delta$ is $(n-1)$-ambidextrous and $(m-1)$-truncated, by induction we have $\delta$ $(m-1)$-ambidextrous, and $\mu_\delta^{(m-1)}$ and $\mu_\delta^{(n-1)}$ are homotopic. Thus the natural transformations $\nu_f^{(m)}$ and $\nu_f^{(n)}$ are homotopic, and since $\nu_f^{(n)}$ is the counit of an adjunction, $\nu_f^{(m)}$ has the same property. $\square$

**Definition 20.1.4.** Let $\mathcal{X}$ be a category with pullbacks, and let $q : \mathcal{C} \to \mathcal{X}$ be a Beck-Chevalley fibration. Then we let $f : X \to Y$ be (weakly) ambidextrous if it is (weakly) $n$-ambidextrous for some $n$. And we let $\nu_f$ and $\mu_f$ denote the $\nu_f^{(n)}$ and $\mu_f^{(n)}$ in the definition. This is well-defined by proposition 20.1.3. In particular, $f$ is ambidextrous iff for every Cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

$f'$ is weakly ambidextrous, and $(f')^*$ admit a right adjoint $f'_*$ isomorphic to its left adjoint $f'_!$. Clearly, ambidextrous morphisms are closed under compositions.

We then state a theorem of maximal generality concerning the naturality and coherence of the construction $f \mapsto \mu_f$ or $\nu_f$. This would use the notion of $(\infty, 2)$-categories, and we will not state a proof of this theorem.

**Theorem 20.1.5.** *Let $q : \mathcal{C} \to \mathcal{X}$ be a Beck-Chevalley fibration. We may associate $q$ to a functor of $(\infty, 2)$-categories $\Theta : \mathcal{Z} \to$ **Cat** (where **Cat** is viewed as an $(\infty, 2)$-category), which may be described imformally as follows.*

1. *$\mathcal{Z}$ has objects the same as the objects of $\mathcal{X}$.*

2. *The 1-morphisms in $\mathcal{Z}$ are spans in $\mathcal{X}$, and the composition is given by the usual composition of spans.*

3. *Given two 1-morphisms in $\mathcal{Z}$, say $X \leftarrow M \to Y$ and $X \leftarrow N \to Y$, a 2-morphism from $M$ to $N$ is given by a span between the 1-morphisms, namely*

$$\begin{tikzcd} & M \arrow[dl] \arrow[dr] & \\ X & P \arrow[l] \arrow[r] \arrow[u, "u"] \arrow[d, "v"'] & Y \\ & N \arrow[ul] \arrow[ur] & \end{tikzcd}$$

*where the map $P \to M$ is ambidextrous.*

4. *To each $X \in \mathcal{Z}$, we let $\Theta(X) = \mathcal{C}_X$.*

5. *To each 1-morphism $X \xleftarrow{f} M \xrightarrow{g} Y$ in $\mathcal{Z}$, $\Theta((f,g)) = g_! f^* : \mathcal{C}_X \to \mathcal{C}_Y$, and the compatibility with respect to the composition of 1-morphisms is given by the Beck-Chevalley condition.*

6. *To each 2-morphism in $\mathcal{Z}$ given by the above diagram, $\Theta$ associates it to the natural transformation*
$$g_! f^* \xrightarrow{\mu_u} g_!(u_! u^*) f^* \simeq g'_!(v_! v^*)(f')^* \xrightarrow{\varepsilon_v} g'_!(f')^*$$

*In particular, we have the commutativity of the following.*
*For a Cartesian square $\tau$*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g_X"'] & Y' \arrow[d, "g_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

*if $f$ is weakly ambidextrous, then the diagram of natural transformation commutes.*

$$\begin{tikzcd} (f')^* f'_! g_X^* \arrow[r, "BC[\tau]"] \arrow[d, "\nu_{f'}"'] & (f')^* g_Y^* f_! \arrow[r] & g_X^* f^* f_! \arrow[d, "\nu_f"] \\ g_X^* \arrow[rr, "\mathrm{id}"'] & & g_X^* \end{tikzcd}$$

*And for $f : X \to Y$, $g : Y \to Z$ weakly ambidextrous, we have $\nu_{gf}$ homotopic to the composition*

$$(gf)^*(gf)_! \simeq f^* g^* g_! f_! \xrightarrow{\nu_g} f^* f_! \xrightarrow{\nu_f} \mathrm{id}_{\mathcal{C}_X}$$

*If $f$ is ambidextrous, then the diagram of natural transformations commutes.*

$$\begin{tikzcd} g_Y^* \arrow[rr, "\mathrm{id}"] \arrow[d, "\mu_{f'}"'] & & g_Y^* \arrow[d, "\mu_f"] \\ f'_!(f')^* g_Y^* \arrow[r] & f'_! g_X^* f^* \arrow[r, "BC[\tau]"'] & g_Y^* f_! f^* \end{tikzcd}$$

*And for $f : X \to Y$, $g : Y \to Z$ ambidextrous, $\mu_{gf}$ is homotopic to the composition*

$$\mathrm{id}_{\mathcal{C}_Z} \xrightarrow{\mu_g} g_! g^* \xrightarrow{\mu_f} g_! f_! f^* g^* \simeq (gf)_!(gf)^*$$

**Definition 20.1.6.** Let $\mathcal{C}$ be a category. Then the construction $X \mapsto \mathrm{Fun}(X, \mathcal{C})$ defines a functor $\mathbf{An}^{\mathrm{op}} \to \mathbf{Cat}$. We let $q : \mathrm{LocSys}(\mathcal{C}) \to \mathbf{An}$ denote the Cartesian fibration classified by this functor. More informally, $\mathrm{LocSys}(\mathcal{C})$ is the category with objects pairs $(X, F)$ where $X \in \mathbf{An}$ is a space and $F : X \to \mathcal{C}$ is a functor (somehow viewed as a locally constant sheaf).

**Theorem 20.1.7.** *Let $\mathcal{C}$ be a category admitting limits and small colimits. Then the forgetful functor* $\mathrm{LocSys}(\mathcal{C}) \to \mathbf{An}$ *is a Beck-Chevalley fibration.*

*Proof.* Firstly notice that given a map $f : X \to Y$ of spaces, we have an adjunction $f_! \dashv f^*$ between the categories $\mathrm{Fun}(X, \mathcal{C})$ and $\mathrm{Fun}(Y, \mathcal{C})$ given by the left Kan extension along $f$ and precomposition with $f$. Then we see that $\mathrm{LocSys}(\mathcal{C}) \to \mathbf{An}$ is indeed a bifibration. Then we verify that it is a Beck-Chevalley fibration. Suppose we have a Cartesian square $\sigma$.

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g_X"'] & Y' \arrow[d, "g_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Unwinding the definitions, we must prove that given local systems $L_X : X \to \mathcal{C}$ and $L_Y : Y \to \mathcal{C}$, and a natural transformation $\alpha : L_X \to L_Y \circ f$ exhibiting $L_Y$ as a left Kan extension of $L_X$ along $f$, then the induced natural transformation $\beta : L_X \circ g_X \to L_Y \circ g_Y \circ f'$ exhibits $L_Y \circ g_Y$ as a left Kan extension of $L_X \circ g_X$ along $f'$.
We use the pointwise colimit formula. Given a point $y' \in Y'$, we wish to prove the canonical map

$$\varinjlim((L_X \circ g_X)|_{X' \times_{Y'} Y'_{/y'}}) \to (L_Y \circ g_Y)(y')$$

is an equivalence. Since $\sigma$ is a Cartesian square, the above map can be identified with the map

$$\varinjlim(L_X|_{X \times_Y Y_{/y}}) \to L_Y(y)$$

where $y = g_Y(y')$. Thus the conclusion follows immediately. $\square$

**Definition 20.1.8.** Let $\mathcal{C}$ be a category having small colimits and let $q : \mathrm{LocSys}(\mathcal{C}) \to \mathbf{An}$ denote the Beck-Chevalley fibration. We say a space $X$ is (weakly) $\mathcal{C}$-ambidextrous if $X \to *$ is (weakly) ambidextrous in the sense of definition 20.1.4 with respect to $q$.

**Definition 20.1.9.** For a Beck-Chevalley fibration $q : \mathcal{C} \to \mathcal{X}$ and a weakly ambidextrous morphism $f : X \to Y$ in $\mathcal{X}$, we define the trace form to be the composition of natural transformations

$$\mathrm{TrFm}_f : f_! f^* f_! f^* \simeq f_!(f^* f_!) f^* \xrightarrow{\nu_f} f_! f^* \xrightarrow{\varepsilon_f} \mathrm{id}$$

**Theorem 20.1.10.** *Let $q : \mathcal{C} \to \mathcal{X}$ be a Beck-Chevalley fibration and let $f : X \to Y$ be a weakly ambidextrous morphism in $\mathcal{X}$. Then the following conditions are equivalent.*

1. *The map $\nu_f : f^* f_! \to \mathrm{id}_{\mathcal{C}_X}$ is the counit of an adjunction $f^* \dashv f_!$.*

2. *The trace form $\mathrm{TrFm}_f : f_! f^* f_! f^* \to \mathrm{id}_{\mathcal{C}_Y}$ exhibits the functor $f_! f^*$ as its own dual in the monoidal category $\mathrm{Fun}(\mathcal{C}_Y, \mathcal{C}_Y)$.*

*Here we recall that $e : X \otimes Y \to \mathbf{1}$ exhibits $X$ as the dual of $Y$ if there exists a map $c : \mathbf{1} \to Y \otimes X$ and both*

$$X \to X \otimes Y \otimes X \to X$$

*and*

$$Y \to Y \otimes X \otimes Y \to Y$$

*is the identity.*

*Proof.* (1) $\implies$ (2): Let $\mu_f : \mathrm{id}_{\mathcal{C}_Y} \to f_! f^*$ be the unit map compatible with $\nu_f$. We construct the map $c : \mathrm{id}_{\mathcal{C}_Y} \to f_! f^* f_! f^*$ by the composition

$$\mathrm{id} \xrightarrow{\mu_f} f_! f^* \xrightarrow{\eta_f} f_! f^* f_! f^*$$

We claim that $c$ and $\mathrm{TrFm}_f$ exhibit $f_! f^*$ as a self-dual object of $\mathrm{Fun}(\mathcal{C}_Y, \mathcal{C}_Y)$. Explicitly, we must verify the two compositions give the identity. Here, we only verify the following, since the other one is done similarly.

$$f_! f^* \xrightarrow{\mathrm{id} \otimes c} f_! f^* f_! \xrightarrow{\mathrm{TrFm} \otimes \mathrm{id}} f_! f^*$$

We have the following commutative diagram

$$\begin{tikzcd}
f_! f^* \arrow[r, "\mathrm{id} \otimes \mu_f"] \arrow[dr, "\mathrm{id} \otimes c"'] & f_! f^* f_! f^* \arrow[r, "\mathrm{id} \otimes \nu_f \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \eta_f \otimes \mathrm{id}"] & f_! f^* \arrow[d, "\mathrm{id} \otimes \eta_f \otimes \mathrm{id}"] \\
& f_! f^* f_! f^* f_! f^* \arrow[r, "\mathrm{id} \otimes \nu_f \otimes \mathrm{id}"'] \arrow[dr, "\mathrm{TrFm}_f \otimes \mathrm{id}"'] & f_! f^* f_! f^* \arrow[d, "\varepsilon_f \otimes \mathrm{id}"] \\
& & f_! f^*
\end{tikzcd}$$

Which immediately implies that the diagonal is the identity. (2) $\implies$ (1): Suppose that we have a coevaluation map $c : \mathrm{id}_{\mathcal{C}_Y} \to f_!f^*f_!f^*$ compatible with $\mathrm{TrFm}_f$. We define natural transformations

$$\mu : \mathrm{id} \xrightarrow{c} f_!f^*f_!f^* \xrightarrow{\varepsilon_f \otimes \mathrm{id}} f_!f^*$$

and

$$\mu' : \mathrm{id} \xrightarrow{c} f_!f^*f_!f^* \xrightarrow{\mathrm{id} \otimes \varepsilon_f} f_!f^*$$

We will prove that the compositions

$$f_! \xrightarrow{\mu \otimes \mathrm{id}} f_!f^*f_! \xrightarrow{\mathrm{id} \otimes \nu_f} f_!$$

and

$$f^* \xrightarrow{\mathrm{id} \otimes \mu'} f^*f_!f^* \xrightarrow{\nu_f \otimes \mathrm{id}} f^*$$

are identities, so that the triangular identities of the pair is satisfied, hence it is an adjoint pair. Again, we only prove the first identity, since the second is similar. Since we have $f_! \dashv f^*$, we have the composite map

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}_Y, \mathcal{C}_Y)}(f_!, f_!) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}_Y, \mathcal{C}_Y)}(f_!f^*, f_!f^*) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}_Y, \mathcal{C}_Y)}(f_!f^*, \mathrm{id}_{\mathcal{C}_Y})$$

is an equivalence. Thus it suffices to verify that

$$f_!f^* \xrightarrow{\mu \otimes \mathrm{id}} f_!f^*f_!f^* \xrightarrow{\mathrm{id} \otimes \nu_f \otimes \mathrm{id}} f_!f^* \xrightarrow{\varepsilon_f} \mathrm{id}_{\mathcal{C}_Y}$$

is homotopic to $\varepsilon_f$. By definitions of TrFm and $\mu$, the composition is equivalent to

$$f_!f^* \xrightarrow{c \otimes \mathrm{id}} f_!f^*f_!f^*f_!f^* \xrightarrow{\varepsilon_f \otimes \mathrm{id}} f_!f^*f_!f^* \xrightarrow{e} \mathrm{id}_{\mathcal{C}_Y}$$

Thus the result follows from the commutativity of the diagram

$$\begin{tikzcd}
f_!f^* \arrow[r, "c\otimes\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & f_!f^*f_!f^*f_!f^* \arrow[r, "\varepsilon_f\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes e"] & f_!f^*f_!f^* \arrow[d, "e"] \\
& f_!f^* \arrow[r, "\varepsilon_f"'] & \mathrm{id}_{\mathcal{C}_Y}
\end{tikzcd}$$

□

## 20.2 The category of spans

**Definition 20.2.1.** We define a full subcategory $\mathcal{C} \to \mathcal{D}$ to be wide if we have $\mathrm{Core}(\mathcal{C}) \to \mathrm{Core}(\mathcal{D})$ is an equivalence.

**Definition 20.2.2.** For a category $\mathcal{C}$, we define a weak coWaldhausen structure on $\mathcal{C}$ to be a wide subcategory $\mathcal{C}^\dagger \to \mathcal{C}$, such that any diagram

$$\begin{tikzcd}
& Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Z
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
W \arrow[r, dashed, "f'"] \arrow[d, dashed, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Z
\end{tikzcd}$$

of the form on the left with $g \in \mathcal{C}^\dagger$, we have a pullback of it having $g' \in \mathcal{C}^\dagger$.

**Definition 20.2.3.** We define the twisted arrow category $\widetilde{\Delta}^n$ to be the nerve of the category consisting of pairs $(i,j) \in [n] \times [n]$ with $i \le j$ and $\mathrm{Hom}((i,j),(i',j'))$ is a singleton for $i' \le i \le j \le j'$ and empty otherwise.
For a weak coWaldhausen category $(\mathcal{C}, \mathcal{C}^\dagger)$, we define a map $f : (\widetilde{\Delta}^n)^{\mathrm{op}} \to \mathcal{C}$ is Cartesian if for every $i' \le i \le j \le j'$, the square

$$\begin{tikzcd} f(i',j') \arrow[r] \arrow[d] & f(i',j) \arrow[d] \\ f(i,j') \arrow[r] & f(i,j) \end{tikzcd}$$

is Cartesian with the vertical morphisms in $\mathcal{C}^\dagger$.
For a weak coWaldhausen category $(\mathcal{C}, \mathcal{C}^\dagger)$, we define the category of spans $\mathrm{Span}(\mathcal{C}, \mathcal{C}^\dagger)$ to be the simplicial set with $n$-simplices the set of Cartesian maps $\widetilde{\Delta}^n \to (\mathcal{C}, \mathcal{C}^\dagger)$.

**Definition 20.2.4.** For $f \in \mathcal{C}^\dagger$, we define $\widehat{f}$ to be the morphism in $\mathrm{Span}(\mathcal{C}, \mathcal{C}^\dagger)$ given by $Y \xleftarrow{f} X \xrightarrow{\mathrm{id}} X$. This is the inverse of $f$ in $Span(\mathcal{C}, \mathcal{C}^\dagger)$.

*Remark* 32. The above definition indeed corresponds to the common construction of spans of 1-categories. In fact, we may identify $\mathrm{Hom}_{\mathrm{Span}(\mathcal{C},\mathcal{C}^\dagger)}(X,Y)$ with the full subgroupoid of $\mathrm{Core}(\mathcal{C}_{/X\times Y})$ spanned by object whose morphism composed with the projection to $X$ lies in $\mathcal{C}^\dagger$.

**Definition 20.2.5.** For a space $X$, we say it is $n$-finite if it is $n$-truncated, and all its homotopy groups of degree below $n$ is finite. We say $X$ is $\pi$-finite if it is $n$-finite for some $n$. We will denote the fullsubcategory of $n$-finite spaces $\mathbf{An}_n$. Then we write $\mathcal{K}_n$ for $\pi_0(\mathrm{Core}(\mathbf{An}_n))$ (that is, we choose a representative for each weak homotopy type and form a set $\mathcal{K}_n$).
We let $\mathbf{An}_{n,m} \subseteq \mathbf{An}_n$ be the wide subcategory containing all $m$-truncated maps. Notice that $(\mathbf{An}_n, \mathbf{An}_{n,m})$ is a coWaldhausen category, hence we let $\mathbf{An}_n^m = \mathrm{Span}(\mathbf{An}_n, \mathbf{An}_{n,m})$ denote the span.
The Cartesian monoidal structure on $\mathbf{An}_n$ induces a symmetric monoidal structure on $\mathbf{An}_n^m$, given by $(X,Y) \mapsto X \times Y$ on the level of objects and by levelwise Cartesian product of spans on the level of morphisms. Notice that this is monoidal structure is not Cartesian on $\mathbf{An}_n^m$.

**Lemma 20.2.6.** *The category $\mathbf{An}_n$ admits $\mathcal{K}_n$-indexed colimits which are preserved and reflected by the inclusion $\mathbf{An}_n \to \mathbf{An}$.*

*Proof.* It suffices to prove that $\mathbf{An}_n$ is closed under $\mathcal{K}_n$-indexed colimits, since the inclusion $\mathbf{An}_n \to \mathbf{An}$ is fully faithful. More explicitly, let $X$ be an $n$-finite Kan complex, and let $\varphi : X \to \mathbf{An}$ be a functor, we show $\varinjlim_X \varphi$ is also $n$-finite. By HTT 3.3.4, we have the identification of the colimit with the fibration $E_\varphi \to X$ classified by $\varphi$. By the long exact sequence of homotopy groups, since both $X$ and the fiber is $n$-finite, we see that $E_\varphi$ is $n$-finite. ☐

**Lemma 20.2.7.** *Let $\mathcal{D}$ be a category which admits $\mathcal{K}_n$-indexed colimits, and let $F : \mathbf{An}_n \to \mathcal{D}$ be a functor. Then $F$ preserves $\mathcal{K}_n$-indexed colimits iff for every $X \in \mathbf{An}_n$, the collection $\{F(i_x) : F(*) \to F(X)\}$ exhibits $F(X)$ as the colimit of the constant $X$-indexed diagram with value $F(*)$.*

*Proof.* $\Longrightarrow$ : The morphisms $\{* \to X\}$ exhibits $X$ as the colimit of the constant functor $*$ with indexing category $X$.
$\Longleftarrow$ : Suppose the converse holds. Let $Y \in \mathcal{K}_n$ be an $n$-finite space, and let $\varphi : Y \to \mathbf{An}_n$. Let $E_\varphi \to Y$ be the fibration classified by $\varphi$. Again by lemma 20.2.6 and HTT 3.3.4, we have $E_\varphi$ is $n$-finite, and $\varphi(y) \to E_\varphi$ exhibits $E_\varphi$ as the colimit of $\varphi$ in $\mathbf{An}_n$. We must show that

$F(\varphi(y)) \to F(E_\varphi)$ exhibits $F(E_\varphi)$ as the colimit of the diagram $\psi = F \circ \varphi : Y \to \mathcal{D}$. Consider the lax commuting diagram

$$\begin{tikzcd}
E_\varphi \arrow[r, "p"] \arrow[dr, "F(*)"'] & Y \arrow[r] \arrow[d, dashed, "\psi"] \arrow[dl, phantom, "\Uparrow"] & * \arrow[dl, "F(E_\varphi)"] \arrow[l, phantom, "\Leftarrow", shift right=3] \\
& \mathcal{D} &
\end{tikzcd}$$

where the left diagonal is the constant functor with value $F(*)$, and the right diagonal sends the point to $F(E_\varphi)$. Now by assumption, we have $\{F(i_z) : F(*) \to F(\varphi(y))\}_{z\in\varphi(y)}$ exhibits $\psi(y) = \varinjlim_{\varphi(y)} F(*)$. Identifying $\varphi(y)$ with the fiber of $p : E_\varphi \to Y$ at $y$ we see that $\psi$ is the left Kan extension of the constant diagram $E_\varphi \to \mathcal{D}$ along $p : E_\varphi \to Y$. Similarly by considering the set of morphisms $\{F(i_z) : F(*) \to F(E_\varphi)\}_{z\in E_\varphi}$ exhibits $F(E_\varphi)$ as the left Kan extension of $F(*)$ along the final morphism $E_\varphi \to *$. Then by the pasting law for Kan extensions, we have $F(E_\varphi)$ a left Kan extension of $\psi$ along $Y \to *$, which is equivalently stating that $F$ preserves this colimit. □

**Lemma 20.2.8.** *For $-2 \le m \le n$, the subcategory inclusion $\mathbf{An}_n \hookrightarrow \mathbf{An}_n^m$ preserves $\mathcal{K}_n$-indexed colimits.*

*Proof.* By lemma 20.2.7, it suffices to show that for every $X \in \mathbf{An}_n^m$, the collection of morphisms $\{i_x : * \to X\}_{x\in X}$ exhibit $X$ as the colimit of the constant diagram $*$ in $\mathbf{An}_n^m$. Equivalently, we need to show that given any test object $Y \in \mathbf{An}_n^m$, the map $\mathrm{Hom}_{\mathbf{An}_n^m}(X, Y) \to \mathrm{Hom}_{\mathbf{An}}(X, \mathrm{Hom}_{\mathbf{An}_n^m}(*, Y))$ is an equivalence of spaces, where the map is given by sending $f : X \to Y$ to the map $x \mapsto i_x^*(f)$, equipped with the compact open topology.
Let $p_X : X \times Y \to X$ denote the projection onto the first space. By remark 32, we have $\mathrm{Hom}_{\mathbf{An}_n^m}(X, Y)$ the full subgroupoid of $\mathrm{Core}((\mathbf{An}_n)_{/X\times Y})$ spanned by the objects $Z \to X \times Y$ where $Z \to X \times Y \to X$ is $m$-truncated. Under this equivalence, the restriction map $i_x^*$ is given by the pullback functor $i_{\{x\}\times Y}^* : (\mathbf{An}_n)_{X\times Y} \to (\mathbf{An}_n)_(\{x\} \times Y)$. By the straightening-unstraightening equivalence, we have the collection of pullback functors $i_{\{x\}\times Y} : \mathbf{An}_{/X\times Y} \to \mathbf{An}_{/Y}$ induces an equivalence $\mathrm{St}_X : \mathbf{An}_{/X\times Y} \simeq \mathrm{Fun}(X, \mathbf{An}_{/Y})$, hence an equivalence after applying Core.
Now, given an object $Z \to X \times Y$ in $\mathrm{Core}(\mathbf{An}_{/X\times Y})$, the condition that $Z \to X \times Y \to X$ is an $m$-truncated map is equivalent to the condition that the essential image of $\mathrm{Core}(\mathrm{St}_X)(Z \to X \times Y) : X \to \mathrm{Core}(\mathbf{An}_{/Y})$ is contained in $\mathrm{Core}((\mathbf{An}_m)_{/Y})$ (by checking the fibers at each $y$). Moreover, since $X$ is $n$-truncated and $m \le n$, so $Z$ is automatically $n$-truncated. Finally, since the morphism from an $n$-truncated space to the point is naturally $m$-truncated, we may identify $\mathrm{Core}((\mathbf{An}_n)_{/Y})$ with $\mathrm{Hom}_{\mathbf{An}_n^m}(*, Y)$, hence we have the desired equivalence of groupoids. □

**Lemma 20.2.9.** *Let $-2 \le m \le n$, then any equivalence in $\mathbf{An}_n^m$ is homotopic to the image of an equivalence in $\mathbf{An}_n \subseteq \mathbf{An}_n^m$.*

*Proof.* Given a span $X \xleftarrow{p} Z \xrightarrow{q} Y$ which is an equivalence in $\mathbf{An}_n^m$, we may associate it to the functor $p_!q^* : \mathrm{Fun}(Y, \mathbf{An}) \to \mathrm{Fun}(X, \mathbf{An})$ (here $q^*$ is given by the precomposition and $p_!$ is given by the left Kan extension). Then, notice that the functor $\mathbf{An}_n^{\mathrm{op}} \to \mathbf{An}$ given by $X \mapsto \mathrm{Fun}(X, \mathbf{An})$ classifies a left fibration over $\mathbf{An}_n$, which is, in fact, a Beck-Chevalley fibration by theorem 20.1.7. This implies that the map sending a span to the map $p_!q^* : \mathrm{Fun}(Y, \mathbf{An}) \to \mathrm{Fun}(X, \mathbf{An})$ respects composition of spans. Thus if the span is an equivalence in $\mathbf{An}_n^m$, then $p_!q^* : \mathrm{Fun}(Y, \mathbf{An}) \to \mathrm{Fun}(X, \mathbf{An})$ is an equivalence of categories. In particular, $p_!q^*$ preserves terminal objects. Since $q^*$ also preserves the terminal object, $p_!(*) : X \to \mathbf{An}$ is the terminal functor. On the other hand, by construction of the left Kan extension, $p_!(*)$ sends $x$ to the homotopy fiber $Z_x$ of $p$ at $x$. Thus $Z_x$ is contractible, hence $p$ is an equivalence. Thus let $r : X \to Z$ be an inverse of $p$. Thus the span is homotopic to the image of $q \circ r \in \mathrm{Hom}_{\mathbf{An}}(X, Y)$. □

**Corollary 20.2.10.** *The subcategory $\mathbf{An}_n \subseteq \mathbf{An}_n^m$ is wide.*

*Proof.* Directly by lemma 20.2.9. □

**Corollary 20.2.11.** *Let $X$ be a space, then any $X$-indexed diagram in $\mathbf{An}_n^m$ comes from an $X$-indexed diagram in $\mathbf{An}_n$.*

*Proof.* Directly by corollary 20.2.10. □

**Corollary 20.2.12.** *For every $-2 \leq m \leq n$, the category $\mathbf{An}_n^m$ admits $\mathcal{K}_n$-indexed colimits. Moreover, $F : \mathbf{An}_n^m \to \mathcal{D}$ preserves $\mathcal{K}_n$ indexed colimits iff the precomposition $\mathbf{An}_n \to \mathbf{An}_n^m \to \mathcal{D}$ preserves $\mathcal{K}_n$ indexed colimits.*

*Proof.* By corollary 20.2.10 and lemma 20.2.8 □

**Theorem 20.2.13.** *The category $\mathbf{An}_n^m$ with the tensor product is a commutative algebra object in $\mathbf{Cat}_{\mathcal{K}_n}$. Here we adopt the notation that $\mathbf{Cat}_{\mathcal{K}_n}$ denotes the category of categories admitting $\mathcal{K}_n$-indexed colimits and the morphisms $\mathcal{K}_n$-indexed colimit preserving functors.*

*Proof.* It suffices to prove that the symmetric monoidal structure on $\mathbf{An}_n^m$ preserves $\mathcal{K}_n$-indexed colimits in each variables, which follows from corollary 20.2.12 and that the symmetric monoidal structure comes from the Cartesian structure on $\mathbf{An}_n$, which clearly preserves colimits in each variables. □

## 20.3 Semiadditivity

**Definition 20.3.1.** For $\mathcal{D}$ a category and $m \geq -2$, we say $\mathcal{D}$ is $m$-semiadditive if $\mathcal{D}$ admits $\mathcal{K}_m$-indexed colimits, and every $m$-finite space is $\mathcal{D}$-ambidextrous in the sense of definition 20.1.8. In short, $m$-semiadditive categories are the categories where $\mathcal{K}_n$-indexed colimits and limits coincide for $n \leq m$.

We have the following characterization of $m$-semiadditivity within $(m-1)$-semiadditive categories.

**Condition 20.3.2.** We consider the following conditions.

1. $\mathcal{D}$ has $\mathcal{K}_n$-indexed colimits.
2. $\mathcal{D}$ is $(m-1)$-semiadditive.
3. $\mathcal{D}$ admits a structure of a $\mathbf{An}_m^{m-1}$-module in $\mathbf{Cat}_{\mathcal{K}_m}$. Explicitly, there is an action of the symmetric monoidal category $\mathbf{An}_m^{m-1}$ on $\mathcal{D}$, such that the action map $\mathbf{An}_m^{m-1} \times \mathcal{D} \to \mathcal{D}$ preserves $\mathcal{K}_m$-indexed colimits in each variable. We denote the action $X \otimes (-) : \mathcal{D} \to \mathcal{D}$ by $[X] : \mathcal{D} \to \mathcal{D}$.

Notice that in the third condition, the action of $X$ on $\mathcal{D}$ is unique. This is since that we may write $X = \varinjlim_X *$, hence $[X] = \varinjlim_X [*]$. Moreover, we may identify $[X]$ with the composed functor $\mathcal{D} \xrightarrow{p^*} \mathrm{Fun}(X, \mathcal{D}) \xrightarrow{p_!} \mathcal{D}$, where $p : X \to *$ is the terminal map.

**Theorem 20.3.3.** *Suppose that $\mathcal{D}$ satisfies the conditions of condition 20.3.2. Then $\mathcal{D}$ is $m$-semiadditive iff the collection of morphisms $[\widehat{i_x}] : [X] \to [*]$ exhibits $[X]$ as the limit in* $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$ *of the constant diagram with value $[*]$.*

**Definition 20.3.4.** Let $X$ be a $m$-finite space. We denote by $\mathrm{Tr}_X : X \times X \to *$ be the morphism in $\mathbf{An}_m^{m-1}$ given by the span

$$X \times X \xleftarrow{\delta} X \to *$$

and its dual span $\widehat{\mathrm{Tr}}_X$ given by

$$* \leftarrow X \xrightarrow{\delta} X \times X$$

**Lemma 20.3.5.** $\mathrm{Tr}_X$ *exhibits* $X$ *as self-dual.*

*Proof.* Obvious (the coevaluation map is given by $\widehat{\mathrm{Tr}}_X$). □

**Lemma 20.3.6.** *Suppose* $\mathcal{D}$ *satisfies condition 20.3.2, and let* $X$ *be an* $m$*-finite space. Then the natural transformation*

$$p_! p^* \simeq [X] \xrightarrow{[p]} [*] \simeq \mathrm{id}$$

*is a counit associated to the adjunction* $p_! \dashv p^*$*. Here* $p$ *is the terminal map* $X \to *$.

*Proof.* If $X$ is empty then $[X]$ is initial in $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$. And thus the counit must be homotopic to $[p]$. We then may assume that $X$ is not empty. In this case, it suffices to prove the natural transformation $[p]^{ad} : p^* \to p^*$ adjoint to $[p] : p_! p^* \to \mathrm{id}$ is an equivalence.
Notice that specifying a natural transformation $T : p^* \to p^*$ is the same as giving an $X$-indexed family of natural transformations $\{T_x : \mathrm{id} \to \mathrm{id}\}_{x\in X}$. Since $\{[i_x] : [*] \to [X]\}$ exhibits $[X]$ as the colimit in $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$, then the adjoint natural transformation of $[p]$, $p^* \to p^*$ is given by the family $\{[p] \circ [i_x] : \mathrm{id} \to \mathrm{id}\}_{x\in X}$. Since each transformation in the family is an equivalence, their colimit is an equivalence. □

**Lemma 20.3.7.** *Suppose* $\mathcal{D}$ *satisfies condition 20.3.2, and let* $X$ *be an* $m$*-finite space. Then the natural transformation*

$$\mathrm{id} \simeq [*] \xrightarrow{[\widehat{p}]} [X] \simeq p_! p^*$$

*is a unit exhibiting* $p_!$ *as the right adjoint of* $p^*$*. In otherwords, it exhibits* $p : X \to *$ *as* $\mathcal{D}$*-ambidextrous.*

*Proof.* By theorem 20.1.10, we have $p_! p^*$ self dual in $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$. Let $\mu_X : \mathrm{id} \to p_! p^*$ be a unit compatible to $\nu_X : p^* p_! \to \mathrm{id}$. We then show that $[\widehat{p}]$ is equivalent to $\mu_X$. Since $p_! p^*$ is self dual, it suffices to compare the natural transformation $p_! p^* \to \mathrm{id}$ dual to $\mu_X$ and $[\widehat{p}]$ respectively.
We first compute the dual of $\mu_X$. Notice that

$$p_! p^* \xrightarrow{\mathrm{id} \otimes \mu_X} p_! p^* p_! p^* \xrightarrow{\mathrm{id} \otimes \nu_X \otimes \mathrm{id}} p_! p^*$$

is homotopic to the identity since $\mu_X$ and $\nu_X$ are the unit and the counit of the adjunction $p^* \dashv p_!$. Thus the dual of $\mu_X$ is the counit $\varepsilon_X : p_! p^* \to \mathrm{id}$.
Then, the action functor $\mathbf{An}_m^{m-1} \to \mathrm{Fun}(\mathcal{D}, \mathcal{D})$ is monoidal and sends $X$ to $[X]$. Then since $X$ is self-dual, the dual of $[\widehat{p}]$ is the image of the dual in $\mathbf{An}_m^{m-1}$. Thus the dual of $[\widehat{p}]$ is $[p] \simeq \varepsilon_X$, which proves the claim. □

**Definition 20.3.8.** Given an $(m-1)$-truncated map $f : X \to Y$ of $m$-finite spaces, we let $\mathrm{St}_f : Y \to \mathbf{An}$ be the straightening of $f$. Since $f$ is $(m-1)$-truncated, the homotopy fiber $X_y$ is $(m-1)$-finite, and hence $\mathrm{St}_f$ factors as $\mathrm{St}_f : Y \to \mathbf{An}_{m-1} \to \mathbf{An}$. Then we let $\overline{\mathrm{St}}_f$ denote the composition $Y \to \mathbf{An}_{m-1} \to \mathbf{An}_m \to \mathbf{An}_m^{m-1}$.

**Definition 20.3.9.** Notice that for any $Y$, we have the pointwise action $\mathrm{Fun}(Y, \mathbf{An}_m^{m-1})$ on $\mathrm{Fun}(Y, \mathcal{D})$. In particular, the map $\overline{\mathrm{St}}_f \in \mathrm{Fun}(Y, \mathbf{An}_m^{m-1})$ gives a functor $[\overline{\mathrm{St}}_f] : \mathrm{Fun}(Y, \mathcal{D}) \to \mathrm{Fun}(Y, \mathcal{D})$. Informally, we have $[\overline{\mathrm{St}}_f](L)(y) = [X_y](L(y))$.

**Lemma 20.3.10.** *Suppose* $\mathcal{D}$ *satisfies condition 20.3.2, then let* $f : X \to Y$ *be an* $(m-1)$*-truncated map of* $m$*-finite spaces. Then there is an equivalence*

$$[\overline{\mathrm{St}}_f] \simeq f_! f^*$$

*of functors* $\mathcal{D}_Y \to \mathcal{D}_Y$.

*Proof.* Consider the base change $g : X \times_Y X \to X$ of $f$ along itself. Then the diagonal map $\delta : X \to X \times_Y X$ is a section of $g$. □

Finish this part!

## 20.4 $\mathbf{An}_n^m$ as a mode

**Theorem 20.4.1.** *Let $-2 \leq m \leq n$ be integers and let $\mathcal{D}$ be an $m$-semiadditive category which admits $\mathcal{K}_n$-indexed colimits. Then the evaluation at $* \in \mathbf{An}_n^m$ induces an equivalence of categories*

$$\mathrm{Fun}_{\mathcal{K}_n}(\mathbf{An}_n^m, \mathcal{D}) \simeq \mathcal{D}$$

*In other words, $\mathbf{An}_n^m$ is the free $m$-semiadditive category admitting $\mathcal{K}_n$-indexed colimits, generated by one point.*

**Definition 20.4.2.** We say a category $\mathcal{C} \in \mathbf{Pr}^{\mathrm{L}}$ is a mode if it is a commutative algebra in $\mathbf{Pr}^{\mathrm{L}}$, and the multiplication map is an equivalence $\mathcal{C} \otimes \mathcal{C} \simeq \mathcal{C}$, here the monoidal structure is of course the Lurie tensor product. Equivalently, $\mathcal{C}$ defines a localization functor $(-) \otimes \mathcal{C}$ onto the category of $\mathcal{C}$-modules.

**Corollary 20.4.3.** *$\mathbf{An}_m^m$ is the mode of $m$-semiadditivity. That is, a presentable category $\mathcal{C}$ is $n$-semiadditive iff $\mathcal{C} \otimes \mathbf{An}_m^m \simeq \mathcal{C}$, or equivalently, iff $\mathcal{C}$ is a $\mathbf{An}_m^m$-module.*

Finish this part!

# Chapter 21

# Semiadditivity

## 21.1 Norms and integrals

**Definition 21.1.1.** Given categories $\mathcal{C}$ and $\mathcal{D}$, we define a normed functor $q : \mathcal{D} \rightarrowtail \mathcal{C}$ to be the data of an adjunction triad $q_! \dashv q^* \dashv q_*$ between $\mathcal{C}$ and $\mathcal{D}$, together with a norm map $\mathrm{Nm}_q : q_! \to q_!$. We say $q$ is iso-normed functor if $\mathrm{Nm}_q$ is an isomorphism.
We write $X_q$ for $q_! q^* X$, and $c_!^q$ $(u_!^q)$ for the counit (unit) of the adjunction $q_! \dashv q^*$, and $c_*^q$ $(u_*^q)$ for the counit (unit) of the adjunction $q^* \dashv q_*$.

**Lemma 21.1.2.** *A normed functor $q$ is iso-normed iff $\mathrm{Nm}_q : q_! \to q_*$ is an isomorphism at $q^* X$ for all $X \in \mathcal{D}$.*

*Proof.* One direction is clear. The other direction follows from the triangular identity of the adjunctions. $\square$

**Lemma 21.1.3.** *A normed functor $q$ is iso-normed iff $c_*^q \circ q^* \mathrm{Nm}_q : q^* q_! \to q^* q_* \to \mathrm{id}$ is a counit map for the adjunction $q^* \dashv q_!$.*

*Proof.* Direct. $\square$

**Definition 21.1.4.** We denote the above counit by $\nu_q$, and its corresponding unit by $\mu_q$.

**Lemma 21.1.5.** *A composition of (iso-) normed functors are (iso-) normed.*

*Proof.* Direct. $\square$

**Definition 21.1.6.** Let $q : \mathcal{D} \rightarrowtail \mathcal{C}$ be an iso-normed functor. For every $X, Y \in \mathcal{C}$, we define an integral map

$$\int_q : \mathrm{Hom}_{\mathcal{D}}(q^* X, q^* Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

given by the composition

$$\mathrm{Hom}_{\mathcal{D}}(q^* X, q^* Y) \xrightarrow{q_*} \mathrm{Hom}_{\mathcal{C}}(q_* q^* X, q_* q^* Y) \xrightarrow{\mathrm{Nm}_q^{-1}} (q_* q^* X, q_! q^* Y) \xrightarrow{c_! \circ - \circ u_*} \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

Or equivalently the composition

$$\mathrm{Hom}_{\mathcal{D}}(q^* X, q^* Y) \xrightarrow{q_!} \mathrm{Hom}_{\mathcal{C}}(q_! q^* X, q_! q^* Y) \xrightarrow{c_! \circ - \circ \mu_q} \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

**Definition 21.1.7.** Let $q : \mathcal{D} \rightarrowtail \mathcal{C}$ be an iso-normed functor. For every $X \in \mathcal{C}$, we define a map $|q|_X : X \to X$ given by $\int_q \mathrm{id}_{q^* X}$. Equivalently, this is given by the $|q| = c_!^q \circ \mu_q$.

The construction satisfies the following naturality conditions.

**Lemma 21.1.8.** *Let $q : \mathcal{D} \rightarrowtail \mathcal{C}$ and let $X, Y, Z \in \mathcal{C}$, then for $f : q^*X \to q^*Y$ and $g : Y \to Z$, we have $g \circ \int_q f = \int_q (q^*g \circ f)$. Conversely, we have for $f : X \to Y$ and $g : q^*Y \to q^*Z$ the equality $\int_q g \circ f = \int_q (g \circ q^*f)$.*

**Lemma 21.1.9.** *For composable iso-normed functors $q \circ p : \mathcal{E} \to \mathcal{D} \to \mathcal{C}$, we have $\int_q \circ \int_p = \int_{q\circ p}$.*

Then we restate a few facts on Beck-Chevalley conditions using our new language.

**Definition 21.1.10.** Consider the following square.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F_{\mathcal{C}}"] \arrow[d, "q^*"'] & \tilde{\mathcal{C}} \arrow[d, "\tilde{q}^*"] \\ \mathcal{D} \arrow[r, "F_{\mathcal{D}}"'] & \tilde{\mathcal{D}} \end{tikzcd}$$

Suppose we have $q_! \dashv q^*$ and $\tilde{q}_! \dashv \tilde{q}^*$, we get a natural transformation

$$BC_![\sigma] : \tilde{q}_! F_{\mathcal{D}} \xrightarrow{u_!^q} \tilde{q}_! F_{\mathcal{D}} q^* q_! \simeq \tilde{q}_! \tilde{q}^* F_{\mathcal{C}} q_! \xrightarrow{c_!^{\tilde{q}}} F_{\mathcal{C}} q_!$$

Dually for $q^* \dashv q_*$ and $\tilde{q}^* \dashv \tilde{q}_*$, we get a natural transformation

$$BC_*[\sigma] F_{\mathcal{C}} q_* \xrightarrow{u_*^{\tilde{q}}} \tilde{q}_* \tilde{q}^* F_{\mathcal{C}} q_* \simeq \tilde{q}_* F_{\mathcal{D}} q^* q_* \xrightarrow{c_*^q} \tilde{q}_* F_{\mathcal{D}}$$

We say $\sigma$ satisfies the $BC_!$ ($BC_*$) property if $q^*$ and $\tilde{q}^*$ admit left (right) adjoints, and the transformation $BC_!$ ($BC_*$) is an isomorphism.

The construction satisfy all possible natural conditions one may expect (for example the passing of squares).

**Definition 21.1.11.** We define a normed square to be a pair of normed functors $q : \mathcal{D} \rightarrowtail \mathcal{C}$ and $\tilde{q} : \tilde{\mathcal{D}} \to \tilde{\mathcal{C}}$ and the following diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F_{\mathcal{C}}"] \arrow[d, "q^*"'] & \tilde{\mathcal{C}} \arrow[d, "\tilde{q}^*"] \\ \mathcal{D} \arrow[r, "F_{\mathcal{D}}"'] & \tilde{\mathcal{D}} \end{tikzcd}$$

We call the square iso-normed if $q$ and $\tilde{q}$ are iso-normed. Then, for a normed square, we have the following norm-diagram

$$\begin{tikzcd} F_{\mathcal{C}} q_! \arrow[r, "\mathrm{Nm}_q"] & F_{\mathcal{C}} q_* \arrow[d, "BC_*"] \\ \tilde{q}_! F_{\mathcal{D}} \arrow[u, "BC_!"] \arrow[r, "\mathrm{Nm}_{\tilde{q}}"'] & \tilde{q}_* F_{\mathcal{D}} \end{tikzcd}$$

Finally, we define a weakly ambidextrous square is a normed square, such that the norm-diagram commutes. An ambidextrous square is a weakly ambidextrous iso-normed square.

Yet again, there are naturality properties.

Then we discuss the interaction of normed functor with the monoidal structure of categories.

**Definition 21.1.12.** Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. Then a $\otimes$-normed functor $q : \mathcal{D} \rightarrowtail \mathcal{C}$ is a normed functor, such that $q^*$ is monoidal (and hence $q_!$ is colax and $q_*$ is lax monoidal), and the composition of the following maps are isomorphisms.

$$q_!(Y \otimes (q^*X)) \to (q_!Y) \otimes (q_!q^*X) \xrightarrow{\mathrm{id} \otimes c_!} (q_!Y) \otimes X$$

$$q_!((q^*X)\otimes Y)\to (q_!q^*X)\otimes(q_!Y)\xrightarrow{c_!\otimes \mathrm{id}} X\otimes(q_!Y)$$

We denote the unit of $\mathcal{C}$ by $1_{\mathcal{C}}$. Then, we define $1_q = q_!q^*1_{\mathcal{C}}$. From the above, we see that $1_q\otimes - \simeq q_!q^*$ by the following.

$$q_!q^*X\simeq q_!(q^*(X)\otimes q^*(1_{\mathcal{C}}))\to X\otimes q_!q^*1_{\mathcal{C}}\simeq X\otimes 1_q$$

Thus we define $(-)_q = q_!q^*(-)$.
Moreover, we have the following map

$$\varepsilon : 1_q\otimes 1_q\simeq q_!q^*q_!q^*1_{\mathcal{C}}\simeq q_!(q^*q_!)q^*1_{\mathcal{C}}\xrightarrow{\nu} q_!q^*1_{\mathcal{C}}\xrightarrow{c_!} 1_{\mathcal{C}}$$

For monoidal categories, iso-normed functors admit a much more accessible characterization.

**Proposition 21.1.13.** *Let $q:\mathcal{D}\rightarrowtail\mathcal{C}$ be a $\otimes$-normed functor of monoidal categories. The following are equivalent.*

1. *$q$ is iso-normed.*
2. *$\mathrm{Nm}_q$ is an isomorphism at $q^*1_{\mathcal{C}}$.*
3. *$\varepsilon : 1_q\otimes 1_q\to 1_{\mathcal{C}}$ is a duality datum.*

*Proof.* This is essentially theorem 20.1.10. □

**Definition 21.1.14.** We define a $\otimes$-normed square to be a pair of $\otimes$-normed functors $q:\mathcal{D}\rightarrowtail\mathcal{C}$ and $\tilde{q}:\tilde{\mathcal{D}}\rightarrowtail\tilde{\mathcal{C}}$ and a commutative square of monoidal category and monoidal functors

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F_{\mathcal{C}}"] \arrow[d, "q^*"'] & \tilde{\mathcal{C}} \arrow[d, "\tilde{q}^*"] \\
\mathcal{D} \arrow[r, "F_{\mathcal{D}}"'] & \tilde{\mathcal{D}}
\end{tikzcd}$$

For such a square we define a colax natural transformation of functors

$$\theta : (-)_{\tilde{q}}F_{\mathcal{C}} = \tilde{q}_!\tilde{q}^*F_{\mathcal{C}}\simeq \tilde{q}_!F_{\mathcal{D}}q^*\xrightarrow{BC_!} F_{\mathcal{C}}q_!q^* = F_{\mathcal{C}}(-)_q$$

Then by the isomorphisms in definition 21.1.12, we have natural isomorphisms

$$L_q : (X\otimes Y)_q\to X_q\otimes Y \qquad\qquad R_q : (X\otimes Y)_q\to X\otimes Y_q$$

**Proposition 21.1.15.** *For a weakly ambidextrous $\otimes$-normed square, if it satisfies the $BC_!$-condition and $q$ is iso-normed, then $\tilde{q}$ is iso-normed, and the $BC_*$-condition holds as well.*

*Proof.* Since $BC_!$-condition is satisfied, the operation $\theta$ is an isomorphism. Notice that $1_{\tilde{q}}\simeq F_{\mathcal{C}}(1_{\mathcal{C}})$ and consider the diagram

$$\begin{tikzcd}
1_{\tilde{q}}\otimes 1_{\tilde{q}} \arrow[r, "L_{\tilde{q}}^{-1}R_{\tilde{q}}^{-1}"] \arrow[d, "\theta\otimes\theta"'] & \tilde{q}_!(\tilde{q}^*\tilde{q}_!)\tilde{q}^*F_{\mathcal{C}}(1_{\mathcal{C}}) \arrow[r, "\tilde{\nu}"] \arrow[dd, "\theta\cdot\theta"'] & \tilde{q}_!\tilde{q}^*F_{\mathcal{C}}(1_{\mathcal{C}}) \arrow[dd, "\theta"'] \arrow[dr, "\tilde{c}_!"] & \\
F_{\mathcal{C}}(1_q)\otimes F_{\mathcal{C}}(1_q) \arrow[d, "\simeq"'] & & & F_{\mathcal{C}}(1_{\mathcal{C}})\simeq 1_{\tilde{\mathcal{C}}} \\
F_{\mathcal{C}}(1_q\otimes 1_q) \arrow[r, "L_q^{-1}R_q^{-1}"'] & F_{\mathcal{C}}(q_!(q^*q_!)q^*1_{\mathcal{C}}) \arrow[r, "\nu"'] & F_{\mathcal{C}}(q_!q^*1_{\mathcal{C}}) \arrow[ur, "c_!"'] &
\end{tikzcd}$$

and $F_{\mathcal{C}}(1_q)\otimes F_{\mathcal{C}}(1_q)\to 1_{\tilde{\mathcal{C}}}$ is a duality datum. By the above diagram, we see that this is also a duality datum for $1_{\tilde{q}}$. Therefore $\tilde{q}$ is iso-normed. Finally, the $BC_*$ condition holds by observing the norm diagram in definition 21.1.11. □

**Definition 21.1.16.** We define an iso-normed functor $q : \mathcal{D} \rightarrowtail \mathcal{C}$ to be amenable if $|q|$ is an isomorphism.

**Lemma 21.1.17.** *For a ambidextrous square*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F_{\mathcal{C}}"] \arrow[d, "q^*"'] & \tilde{\mathcal{C}} \arrow[d, "\tilde{q}^*"] \\ \mathcal{D} \arrow[r, "F_{\mathcal{D}}"'] & \tilde{\mathcal{D}} \end{tikzcd}$$

*such that $F_{\mathcal{C}}$ is conservative, if $\tilde{q}$ is amenable then $q$ is amenable.*

*Proof.* Since the square is ambidextrous, we have $F_{\mathcal{C}}(|q|_{(-)}) = |\tilde{q}|_{F_{\mathcal{C}}(-)}$. By conservativity of $F_{\mathcal{C}}$ we obtain the results immediately. $\square$

**Lemma 21.1.18.** *Let $q : \mathcal{D} \rightarrowtail \mathcal{C}$ be an iso-normed functor, if $q$ is amenable, then for every $X \in \mathcal{C}$ the counit $c_! : q_! q^* X \to X$ has a section up to homotopy. In particular, every object of $\mathcal{C}$ is the retract of an object in the image of $q_!$.*

*Proof.* By the following identity.

$$|q|_X = \int_q \mathrm{id}_{q^*X} = \int_q (q^*(c_!)_X \circ (u_!)_{q^*X}) = (c_!)_X \circ \int_q (u_!)_{q^*X}$$

Thus if $|q|_X$ is an isomorphism, then $(c_!)_X$ has a section. $\square$

**Lemma 21.1.19.** *Let $\mathcal{E} \xrightarrow{p} \mathcal{D} \xrightarrow{q} \mathcal{C}$ be a pair of normed functors. If $p$ is amenable and $qp$ is iso-normed, then $q$ is iso-normed.*

*Proof.* Notice we have

$$\mathrm{Nm}_{qp} : q_! p_! \xrightarrow{\mathrm{Nm}_p} q_* p_! \xrightarrow{\mathrm{Nm}_q} q_* p_*$$

since $\mathrm{Nm}_{qp}$ and $\mathrm{Nm}_p$ are isomorphisms, so is $\mathrm{Nm}_q : q_! p! \to q_* p_!$. But every $X \in \mathcal{D}$ is a retract of an element of the form $p_! Y$ for $Y \in \mathcal{E}$, hence since isomorphisms are closed under retracts, we have $\mathrm{Nm}_q$ an isomorphism for every $X \in \mathcal{D}$. $\square$

## 21.2 Application to local systems

**Definition 21.2.1.** Recall from theorem 20.1.7 we have $\mathrm{LocSys}(\mathcal{C}) \to \mathbf{An}$ a Beck-Chevalley fibration. Moreover, recall the definition of (weakly) $\mathcal{C}$-ambidextrous morphisms from definition 20.1.4. By the definition given in the last section, we have for any weakly $\mathcal{C}$-ambidextrous morphism $q : A \to B$ is associated to a normed functor

$$q^c_{\mathcal{C}} : \mathrm{Fun}(A, \mathcal{C}) \rightarrowtail \mathrm{Fun}(B, \mathcal{C})$$

with $(q^c_{\mathcal{C}})^* = q^*$ and other morphisms of definition likewise defined. By definition, $q^c_{\mathcal{C}}$ is iso-normed iff $q$ is $\mathcal{C}$-ambidextrous.
Then recall the definition of amenability from definition 21.1.16. We say $q : A \to B$ is amenable if it is $\mathcal{C}$-ambidextrous and $q^c_{\mathcal{C}}$ is amenable.
In the special case where $q : A \to *$, $|q|$ is a natural transformation $|q| : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$, and we also denote this by $|A|_{\mathcal{C}}$, called the $\mathcal{C}$-cardinality of $A$.

From now on, as in the previous chapter, we abbreviate the terminal morphism of spaces $!_X : X \to *$ by $X$, and adopt this convention to any notation involved.

**Example 9.** *Notice that for $q : [n] \to *$, we have $q_!$ the coproduct of $n$ objects, and $q_*$ the product of $n$ objects. Moreover, if $\mathcal{C}$ is 0-semiadditive, $|q|_X$ is given by the multiplication by $n$ map.*

**Construction 21.2.2.** Recall the notion of Beck-Chevalley fibrations in definition 20.1.1, and recall that it coincides with the $BC_!$ property in definition 21.1.10. This actually implies the following. Consider the following pullback square

$$\begin{tikzcd}
\tilde{A} \arrow[r, "s_A"] \arrow[d, "\tilde{q}"'] \arrow[dr, phantom, "\ulcorner", very near start] & A \arrow[d, "q"] \\
\tilde{B} \arrow[r, "s_B"'] & B
\end{tikzcd}$$

which induces the base change square

$$\begin{tikzcd}
\mathrm{Fun}(B, \mathcal{C}) \arrow[r, "s_B^*"] \arrow[d, "q^*"'] & \mathrm{Fun}(\tilde{B}, \mathcal{C}) \arrow[d, "\tilde{q}^*"] \arrow[dl, phantom, "\lrcorner", very near end] \\
\mathrm{Fun}(A, \mathcal{C}) \arrow[r, "s_A^*"'] & \mathrm{Fun}(\tilde{A}, \mathcal{C})
\end{tikzcd}$$

From theorem 20.1.7 we see that this square admits the $BC_!$ condition and $BC_*$ condition iff $\mathcal{C}$-admits colimits or limits of the shape $q^{-1}(b)$ for all $b$. This gives rise to the following naturality condition on the integral. Suppose $q$ is ambidextrous, then for all $X, Y \in \mathrm{Fun}(B, \mathcal{C})$, and $f : q^*X \to q^*Y$, we have

$$s_B^* \int_q f \simeq \int_{\tilde{q}} s_A^* f$$

by the second definition of the integral. Consequently, we also have

$$s_B^* |q|_X = |\tilde{q}|_{s_B^* X}$$

directly.
Not that directly, we have the associativity for integrals. That is, consider the pullback square

$$\begin{tikzcd}
A_2 \times_B A_1 \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, "q_2 \times_B q_1"] & A_1 \arrow[d, "q_1"] \\
A_2 \arrow[r, "q_2"'] & B
\end{tikzcd}$$

such that $q_1$ and $q_2$ are $\mathcal{C}$-ambidextrous, then so is $\pi_1$, $\pi_2$ and the diagonal. We have for all $X, Y, Z \in \mathrm{Fun}(B, \mathcal{C})$ and maps $f_1 : q_1^* X \to q_1^* Y$, $f_2 : q_2^* Y \to q_2^* Z$, an equivalence

$$\int_{q_2 \times_B q_1} (\pi_2^* f_2 \circ \pi_1^* f_1) \simeq \int_{q_2} f_2 \circ \int_{q_1} f_1$$

In particular, we have

$$|q_2 \times_B q_1|_X \simeq |q_2|_X \circ |q_1|_X$$

Amenable spaces are principally detected and used in the following ways.

**Corollary 21.2.3.** *Consider the above pullback square of spaces. If $\pi_0(s_B)$ is surjective and $\tilde{q}$ is $\mathcal{C}$-amenable, then $q$ is $\mathcal{C}$-amenable.*

*Proof.* Follows from lemma 21.1.17. □

**Proposition 21.2.4.** *Let $A \to E \to B$ be a fiber sequence of weakly $\mathcal{C}$-ambidextrous spaces and $B$ is connected. Then if $A$ is amenable and $E$ is ambidextrous, then $B$ is also ambidextrous.*

*Proof.* By lemma 21.1.19. □

**Proposition 21.2.5.** *Let $A$ be a connected space. If $\Omega A$ is $\mathcal{C}$-amenable, then the counit map $c_!^A : A_! A^* \to \mathrm{id}$ is an isomorphism.*

*Proof.* Consider a choice of basepoint $e : * \to A$. Then the composition

$$\mathrm{id} \simeq A_! e_! e^* A^* \xrightarrow{\mathrm{id} \circ c_!^e \circ \mathrm{id}} A_! A^* \xrightarrow{c_!^A} \mathrm{id}$$

is the counit of the adjunction

$$\mathrm{id} \simeq A_! e_! \rightleftarrows e^* A^* \simeq \mathrm{id}$$

Hence $A_! c_!^e A^*$ is a left inverse of $c_!^A$, and to show $c_!^A$ is an isomorphism it suffices to show that $A_! c_!^e A^*$ thus $c_!^e$ admit a further left inverse. By corollary 21.2.3 $e$ is amenable, by lemma 21.1.18 we obtain the result. □

**Definition 21.2.6.** We define a functor $F : \mathcal{C} \to \mathcal{D}$ between $m$-semiadditive categories to be $m$-semiadditive if it preserves $m$-finite colimits.

**Definition 21.2.7.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of categories and $q : A \to B$ a map of spaces. We define the $(F, q)$-square to be the following commutative square of categories.

$$\begin{tikzcd} \mathrm{Fun}(B, \mathcal{C}) \arrow[r, "F_*"] \arrow[d, "q^*"'] & \mathrm{Fun}(B, \mathcal{D}) \arrow[d, "q^*"] \\ \mathrm{Fun}(A, \mathcal{C}) \arrow[r, "F_*"'] & \mathrm{Fun}(A, \mathcal{D}) \end{tikzcd}$$

**Lemma 21.2.8.** *Like the above, if $\mathcal{C}$ and $\mathcal{D}$ admit and $F$ preserves all $q^{-1}(b)$ shaped colimits (limits), then the $(F, q)$-square satisfies the $BC_!$ ($BC_*$) condition.*

*Proof.* By the pointwise construction of Kan extensions used in the proof of theorem 20.1.7. □

We have the following categorical result.

**Theorem 21.2.9.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor preserving $(m-1)$-finite colimits. Let $q : A \to B$ be a $m$-finite map of spaces. Then if $q$ is (weakly) $\mathcal{C}$-ambidextrous and (weakly) $\mathcal{D}$-ambidextrous, then the $(F, q)$-square is (weakly) ambidextrous.*

*Proof.* Recall that a weakly ambidextrous square is ambidextrous iff $q_{\mathcal{C}}^c$ and $q_{\mathcal{D}}^c$ is iso-normed, hence it suffices to prove the weakly ambidextrous part of the claim, that is, the norm-diagram in definition 21.1.11 commutes.
Firstly, notice that the commutativity of the norm diagram

$$\begin{tikzcd} F_{\mathcal{C}} q_! \arrow[r, "\mathrm{Nm}_q"] & F_{\mathcal{C}} q_* \arrow[d, "BC_*"] \\ \tilde{q}_! F_{\mathcal{D}} \arrow[u, "BC_!"] \arrow[r, "\mathrm{Nm}_{\tilde{q}}"'] & \tilde{q}_* F_{\mathcal{D}} \end{tikzcd}$$

is equivalent to the commutativity of the following wrong way counit diagram

$$\begin{tikzcd} \tilde{q}^* F_{\mathcal{C}} q_! \arrow[r, "\simeq"] & F_{\mathcal{D}} q^* q_! \arrow[d, "\nu_q"] \\ \tilde{q}^* \tilde{q}_! F_{\mathcal{D}} \arrow[u, "BC_!"] \arrow[r, "\nu_{\tilde{q}}"'] & F_{\mathcal{D}} \end{tikzcd}$$

by moving the $\tilde{q}_*$ on the bottom right to the left functors by adjunction. To prove that the second diagram commutes, we consider the following diagram of spaces.

$$\begin{tikzcd}
A \arrow[dr, "\delta"] \arrow[drr, bend left, "\simeq"] \arrow[ddr, bend right, "\simeq"'] & & \\
& A \times_B A \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & A \arrow[d, "q"] \\
& A \arrow[r, "q"'] & B
\end{tikzcd}$$

Since both $\mathcal{C}$ and $\mathcal{D}$ admits $m$-finite colimits, we have by **??** that the natural transformation

$$BC_! : (\pi_2)_! \pi_1^* \to q^* q_!$$

is an isomorphism for local systems with coefficients in $\mathcal{C}$ or $\mathcal{D}$. Then by definition, $\nu_q^{\mathcal{C}}$ is the composition

$$q^* q_! \xrightarrow{BC_!^{-1}} (\pi_2)_! \pi_1^* \xrightarrow{\mu_\delta} (\pi_2)_! \delta_! \delta^* \pi_1^* \simeq \mathrm{id}$$

where $\mu_\delta$ exists since by definition $q$ is weakly ambidextrous hence $\delta$ is ambidextrous. Therefore we only need to prove the commutativity of each the small part in the following diagram (notice the source and target of each functor in the following diagram is not written out explicitly, but it is a functor between local systems with coefficients in $\mathcal{C}$ before composing with $F_*$ and with coefficients in $\mathcal{D}$ after composing with $F_*$).

$$\begin{tikzcd}
q^* q_! F_* \arrow[r, "BC_!^{\mathcal{D}}"] \arrow[dd, "BC[q]_!"'] & (\pi_2)_! \pi_1^* F_* \arrow[r, "\mu_\delta^{\mathcal{D}}"] \arrow[dd, "\simeq"'] & (\pi_2)_! \delta_! \delta^* \pi_1^* F_* \arrow[d, "\simeq"] \arrow[ddr, bend left, "\simeq"] & \\
& & (\pi_2)_! \delta_! \delta^* F_* \pi_1^* \arrow[d, "\simeq"] & \\
q^* F_* q_! \arrow[dd, "\simeq"'] & (\pi_2)_! F_* (\pi_1^*) \arrow[ur, "\mu_\delta^{\mathcal{D}}"] \arrow[dr, "\mu_\delta^{\mathcal{C}}"'] \arrow[dd, "BC[\pi_2]_!"'] & (\pi_2)_! \delta_! F_* \delta^* \pi_1^* \arrow[r, "\simeq"] \arrow[d, "BC[\delta]_!"] & F_* \\
& & (\pi_2)_! F_* \delta_! \delta^* \pi_1^* \arrow[d, "BC[\pi_2]_!"'] & \\
F_* q^* q_! \arrow[r, "BC_!^{\mathcal{C}}"'] & F_* (\pi_2)_! \pi_1^* \arrow[r, "\mu_\delta^{\mathcal{C}}"'] & F_* (\pi_2)_! \delta_! \delta^* \pi_1^* \arrow[uur, bend right] &
\end{tikzcd}$$

Where in the diagram $BC[f]_!$ denote the Beck-Chevalley transformation for the square

$$\begin{tikzcd}
\mathrm{Fun}(B, \mathcal{C}) \arrow[r, "F_*"] \arrow[d, "f^*"'] & \mathrm{Fun}(B, \mathcal{D}) \arrow[d, "f^*"] \\
\mathrm{Fun}(A, \mathcal{C}) \arrow[r, "F_*"'] & \mathrm{Fun}(A, D)
\end{tikzcd}$$

Then all the small patches of the diagram commutes, hence the whole diagram commutes, which proves the claim. $\square$

This strong technical result immediately gives the following corollaries.

**Corollary 21.2.10.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of $m$-semiadditive categories. Then the functor $F$ preserves $m$-finite colimits iff it preserves $m$-finite limits.*

*Proof.* We prove by induction on $m$. The assertion is obvious when $m = -2$, and then we assume the claim holds for $m-1$. Then we have $F$ preserves both $(m-1)$-finite limits and colimits. Then for every $m$-finite space $A$, consider $A : A \to *$. Since $\mathcal{C}$ and $\mathcal{D}$ are $(m-1)$-semiadditive and $F$ preserves $(m-1)$-finite colimits, we have the $(F, A)$-square weakly ambidextrous. Since $\mathcal{C}$ and $\mathcal{D}$ are $m$-semiadditive, we have the square ambidextrous. Thus the $(F, A)$-square satisfies the $BC_!$ condition iff it satisfies the $BC_*$ condition. Explicitly, $F$ preserves $A$ colimits iff it preserves $A$ limits. □

**Corollary 21.2.11.** *Let $\{\mathcal{C}_i\}$ be a collection of $m$-semiadditive categories. Then $\mathcal{C} = \prod \mathcal{C}_i$ is $m$-semiadditive.*

*Proof.* We again prove by induction on $m$. Again the assertion is obvious when $m = -2$. Then let $q : A \to B$ be an $m$-finite space. Then by induction, $\mathcal{C}$ is $(m-1)$-semiadditive, hence $q$ is weakly $\mathcal{C}$-ambidextrous. Therefore it suffices to show that $\mathrm{Nm}_q^{\mathcal{C}}$ is an isomorphism. By theorem 21.2.9, $q$ is $\mathcal{C}_i$-ambidextrous and the projection $\pi : \mathcal{C} \to \mathcal{C}_i$ preserves colimits, hence the $(\pi_i, q)$-square is weakly ambidextrous. Also, since $\pi_i$ commutes with limits and colimits, the $(\pi_i, q)$-square satisfies both $BC_!$ and $BC_*$ conditions. Hence we have $\pi_i q_! \to \pi_i q_*$ is an isomorphism, and by the jointly conservativity of $\pi_i$ we have $q_! \to q_*$ an isomorphism. □

**Corollary 21.2.12.** *Let $F : \mathcal{C} \to \mathcal{D}$ be an $m$-semiadditive functor and let $q : A \to B$ be an $m$-finite map of spaces. For all $X, Y \in \mathrm{Fun}(B, \mathcal{C})$ and $f : q^*X \to q^*Y$, we have*

$$F(\int_q f) \simeq \int_q F(f) \in \mathrm{Hom}_{\mathrm{Fun}(B,\mathcal{D})}(FX, FY)$$

*In particular $F(|q|_X) = |q|_{F(X)}$.*

*Proof.* The $(F, q)$-square is ambidextrous by theorem 21.2.9, and satisfies the $BC$ conditions by theorem 20.1.7. Thus by the naturality conditions of the normed squares we obtain the results. □

As before, the theory can be slightly simplified in symmetric monoidal categories due to the presence of tensor units.

**Definition 21.2.13.** An $m$-semiadditively symmetric monoidal category is an $m$-semiadditive symmetric monoidal category such that the tensor product distribute over $m$-finite colimits.

**Theorem 21.2.14.** *Let $\mathcal{C}$ be a symmetric monoidal category. Let $q : A \to B$ be a weakly $\mathcal{C}$-ambidextrous map of spaces such that the tensor product of $\mathcal{C}$ distribute over colimits of the form $q^{-1}(b)$. Then the normed functor*

$$q_{\mathcal{C}}^c : \mathrm{Fun}(A, \mathcal{C}) \rightarrowtail \mathrm{Fun}(B, \mathcal{C})$$

*is $\otimes$-normed in a canonical way.*

*Proof.* Direct by checking the conditions in definition 21.1.12. □

**Corollary 21.2.15.** *Let $F : \mathcal{C} \to \mathcal{D}$ be an $m$-finite colimit preserving monoidal functor between monoidal categories admitting $m$-finite colimits and the tensor product distribute over the $m$-finite colimits. Then if $\mathcal{C}$ is $m$-semiadditive, so is $\mathcal{D}$.*

*Proof.* Since $\mathbf{An}_m^m$ is a mode, any $m$-finite colimit preserving functor $F : \mathcal{C} \to \mathcal{D}$ is a map of algebras in $\mathbf{Cat}$, hence preserves the property of being classified by the $\mathbf{An}_m^m$ mode. Thus $\mathcal{C}$ is $m$-semiadditive implies $\mathcal{D}$ is $m$-semiadditive. □

**Lemma 21.2.16.** *Let $\mathcal{C}$ be a $m$-semiadditively monoidal category with unit $1$ and $A$ an $m$-finite space. Then we have for every $X \in \mathcal{C}$, $|A|_X \simeq \mathrm{id}_X \otimes |A|_1$. In particular, $A$ is $\mathcal{C}$-amenable iff $|A|_1$ is an isomorphism. Therefore, we will later denote $|A|_1$ by $|A|$ (which is a morphism $|A| : 1 \to 1$).*

*Proof.* Notice given an object $X$, the functor $- \otimes X : \mathcal{C} \to \mathcal{C}$ preserves $m$-finite colimits. Thus the integral distribute over $F_X$, and we have

$$\mathrm{id}_X \otimes |A|_1 \simeq (- \otimes X)(|A|_1) \simeq |A|_X$$

□

**Lemma 21.2.17.** *$|A|_* \in \mathrm{Hom}_{\mathbf{An}_m^m}(*, *)$ is given by the span $* \leftarrow A \to *$.*

*Proof.* Recall from lemma 20.3.6 and lemma 20.3.7 that for an action of $\mathbf{An}_m^m$ on an $m$-semiadditive category $\mathcal{C}$, if we denote for a $m$-finite space $X$ the action of $X$ on $\mathcal{C}$ by $[X] : \mathcal{C} \to \mathcal{C}$, we have the counit for the adjunction $[X] \to \mathrm{id}$ given by the span $X : X \leftarrow X \to *$ (hence a natural transformation $[X] \to [*]$ given by the action functor $\mathbf{An}_m^m \to \mathrm{End}(\mathcal{C})$), and the wrong way counit exhibiting the ambidexterity the dual span $X^\vee : * \leftarrow X \to X$. Therefore, as the composition of the two, $|X|_*$ is the span computed as follows.

$$\begin{tikzcd}
 & & A \arrow[dl] \arrow[dr] & & \\
 & A \arrow[dl] \arrow[dr] & & A \arrow[dl] \arrow[dr] & \\
* & & A & & *
\end{tikzcd}$$

□

**Lemma 21.2.18.** *Let $\mathcal{C}$ be a symmetric monoidal category. For every $\mathcal{C}$-ambidextrous space such that $\otimes$ distribute over $A$-colimits, the object $1_A$ (recall this is the object given by $A_! A^* 1 \in \mathrm{Fun}(*, \mathcal{C}) \simeq \mathcal{C}$) is dualizable.*

*Proof.* By proposition 21.1.13 directly. □

**Definition 21.2.19.** Let $\mathcal{C}$ be a symmetric monoidal category with tensor unit $1$. Then for any dualizable object $X \in \mathcal{C}$ with dual $X^\vee$, we define the symmetric monoidal dimension of $X$ $\dim_{\mathcal{C}}(X)$ to be a morphism $1 \to 1$ given by $1 \xrightarrow{\eta} X \otimes X^\vee \to X^\vee \otimes X \xrightarrow{\varepsilon} 1$, where $\eta$ is the coevaluation map and $\varepsilon$ is the evaluation map. Obviously, symmetric monoidal dimensions are preserved by symmetric monoidal functors.
Then for a space $A$, we say $A$ is dualizable in $\mathcal{C}$ if $1_A$ is dualizable, and we write $\dim_{\mathcal{C}}(A) = \dim_{\mathcal{C}}(1_A)$.

**Lemma 21.2.20.** *Every $m$-finite space $A$ is self dual in $\mathbf{An}_m^m$ with symmetric monoidal dimension $\dim_{\mathbf{An}_m^m}(A) = (* \leftarrow A^{S^1} \to *) = |A^{S^1}| \in \mathrm{Hom}_{\mathbf{An}_m^m}(*, *)$.*

*Proof.* Notice that we have $\varepsilon : (A \times A \xleftarrow{\Delta} A \to *)$ and $\eta(* \leftarrow A \xrightarrow{\Delta} A \times A)$ are the duality data of $A$ by direct checking. Thus the symmetric monoidal dimension is computed by the following diagram.

$$\begin{tikzcd}
 & & A^{S^1} \arrow[dl] \arrow[dr] & & \\
 & A \arrow[dl] \arrow[dr] & & A \arrow[dl] \arrow[dr] & \\
* & & A \times A & & *
\end{tikzcd}$$

and by lemma 21.2.17, we have $(* \leftarrow A^{S^1} \to *) = |A^{S^1}|$. $\square$

**Corollary 21.2.21.** *Let $\mathcal{C}$ be an $m$-semiadditively symmetric monoidal category with tensor unit $1_{\mathcal{C}}$. Then we have $\dim_{\mathcal{C}}(A) = |A^{S^1}|$. If furthermore $A$ is a loopspace, we have $\dim_{\mathcal{C}}(A) = |A||\Omega A|$.*

*Proof.* Firstly by corollary 21.2.12 and the fact that symmetric monoidal dimension is preserved by symmetric monoidal functors, we have

$$F(\dim_{\mathbf{An}_m^m} A) \simeq F(|A^{S^1}|) \simeq |A^{S^1}|_{F(*)} \simeq |A^{S^1}|_{1_{\mathcal{C}}} \simeq \dim_{\mathcal{C}}(A)$$

Then if $A$ is a loopspace, $A^{S^1} \simeq A \times \Omega A$ splits by simple arguments, and by construction 21.2.2 we obtain

$$\dim_{\mathcal{C}}(A) = |A^{S^1}| = |A \times \Omega A| = |A| \circ |\Omega A|$$

$\square$

## 21.3 Arithmetic aspects

**Definition 21.3.1.** Notice for any object in any (reasonably well-behaved) category $X \in \mathcal{C}$, we define the wreath operation $X \wr C_p$ by $X^p_{hC_p}$, where the $C_p$ action on $X^p$ is given by permuting the factors.

**Lemma 21.3.2.** *The functor $(-) \wr C_p : \mathbf{An} \to \mathbf{An}$ preserves fiber products.*

*Proof.* We let $e : * \to BC_p$ be a choice of a base point, then notice $(-) \wr C_p$ is equivalent to the composition

$$\mathbf{An} \xrightarrow{e_*} \mathrm{Fun}(BC_p, \mathbf{An}) \simeq \mathbf{An}_{/BC_p} \xrightarrow{\pi} \mathbf{An}$$

Then $e_*$ is a right adjoint, hence preserves limits. Also, $\pi$ preserves limits of contractible shape. $\square$

**Definition 21.3.3.** Let $\mathcal{C}$ be a symmetric monoidal category. Consider $(-) \wr C_p : \mathbf{Cat} \to \mathbf{Cat}$. Explicitly, we have a functor $(-) \wr C_p : \mathrm{Fun}(A, \mathcal{C}) \to \mathrm{Fun}(A \wr C_p, \mathcal{C} \wr C_p)$. Then since $\bigotimes : \mathcal{C}^p \to \mathcal{C}$ is symmetric monoidal, we have it factoring through $\bigotimes : (\mathcal{C}^p)_{hC_p} \to \mathcal{C}$. Then we define the functor $\Theta^p_A$ to be the composition

$$\mathrm{Fun}(A, \mathcal{C}) \xrightarrow{(-)\wr C_p} \mathrm{Fun}(A \wr C_p, \mathcal{C} \wr C_p) \xrightarrow{\otimes} \mathrm{Fun}(A \wr C_p, \mathcal{C})$$

**Construction 21.3.4.** The construction $\Theta^p_A$ admits the two following naturality conditions. For a symmetric monoidal functor $F : \mathcal{C} \to \mathcal{D}$, we have

$$\begin{tikzcd} \mathrm{Fun}(A, \mathcal{C}) \arrow[r, "\Theta^p_A"] \arrow[d, "F_*"'] & \mathrm{Fun}(A \wr C_p, \mathcal{C}) \arrow[d, "F_*"] \\ \mathrm{Fun}(A, \mathcal{D}) \arrow[r, "\Theta^p_A"'] & \mathrm{Fun}(A \wr C_p, \mathcal{D}) \end{tikzcd}$$

For a map of spaces $q : A \to B$, we have

$$\begin{tikzcd} \mathrm{Fun}(B, \mathcal{C}) \arrow[r, "\Theta^p_B"] \arrow[d, "q^*"'] & \mathrm{Fun}(B \wr C_p, \mathcal{C}) \arrow[d, "(q \wr C_p)^*"] \\ \mathrm{Fun}(A, \mathcal{C}) \arrow[r, "\Theta^p_A"'] & \mathrm{Fun}(A \wr C_p, \mathcal{C}) \end{tikzcd}$$

We call the second square the $\Theta^p$-square of $q$. Thus if $q$ is finite, so is $q \wr C_p$, hence if $\mathcal{C}$ is $(m-1)$-semiadditive and has $m$-finite colimits, then the $\Theta^p$-square is canonically normed, and is iso-normed if $\mathcal{C}$ is $m$-semiadditive.

**Corollary 21.3.5.** *Again, we have for a map $q : A \to B$ and $\mathcal{C}$ a symmetric monoidal category admitting all (co)limits of the shape $q^{-1}(b)$, and that the tensor product distributes over all $q$-(co)limits, then the $\Theta^p$-square of $q$ satisfies the $BC_*$ ($BC_!$) condition.*

*Proof.* We consider

$$\begin{tikzcd} \mathrm{Fun}(B,\mathcal{C}) \arrow[r, "\Theta^p_B"] \arrow[d, "q^*"'] & \mathrm{Fun}(B \wr C_p,\mathcal{C}) \arrow[r, "\pi_B^*"] \arrow[d, "(q\wr C_p)^*"] & \mathrm{Fun}(B^p,\mathcal{C}) \arrow[d, "(q^p)^*"] \\ \mathrm{Fun}(A,\mathcal{C}) \arrow[r, "\Theta^p_A"'] & \mathrm{Fun}(A \wr C_p,\mathcal{C}) \arrow[r, "\pi_A^*"'] & \mathrm{Fun}(A^p,\mathcal{C}) \end{tikzcd}$$

Then the right square satisfies the $BC$-conditions by theorem 20.1.7, and since $\pi_B^*$ is conservative, it suffices to prove the outer square satisfies the $BC_*$ ($BC_!$) condition, which is obvious. □

**Theorem 21.3.6.** *Let $\mathcal{C}$ be a $m$-semiadditively symmetric monoidal category and let $q : A \to B$ be an $m$-finite map of spaces, then the $\Theta^p$ square of $q$ is ambidextrous.*

*Proof.* The proof is similar to theorem 21.2.9. □

**Theorem 21.3.7.** *Let $\mathcal{C}$ be an $m$-semiadditively symmetric monoidal category and $q : A \to B$ an $m$-finite map of spaces. Then for any $X, Y \in \mathrm{Fun}(B,\mathcal{C})$ and $f : q^*X \to q^*Y$, we have*

$$\Theta^p_B\Big(\int_q f\Big) \simeq \int_{q\wr C_p} \Theta^p_A(f) \in \mathrm{Hom}_{\mathrm{Fun}(B\wr C_p,\mathcal{C})}(\Theta^p_B X, \Theta^p_B Y)$$

*Proof.* By corollary 21.3.5 the $\Theta^p$ squares satisfies the $BC$ conditions, and by theorem 21.3.6 we have it ambidextrous, hence by corollary 21.2.12 the theorem follows immediately. □

Then we present the key property of $\Theta^p$ exhibiting its arithmetic essence. Namely, we calculate the difference between $\Theta^p(f+g)$ and $\Theta^p(f)+\Theta^p(g)$, and find that $\Theta^p$ would be an additive derivation (we will define this later).

To calculate this, we must investigate the $\Theta^p$-square for $q : * \sqcup * \to *$.

**Construction 21.3.8.** The $\Theta^p$-square can be written as

$$\begin{tikzcd} \mathrm{Fun}(*,\mathcal{C}) \arrow[r, "\Theta^p_*"] \arrow[d, "q^*"'] & \mathrm{Fun}(BC_p,\mathcal{C}) \arrow[d, "(q\wr C_p)^*"] \\ \mathrm{Fun}(* \sqcup *,\mathcal{C}) \arrow[r, "\Theta^p_{*\sqcup *}"'] & \mathrm{Fun}((* \sqcup *) \wr C_p,\mathcal{C}) \end{tikzcd}$$

Firstly, the left arrow $q^*$ can be identified with the diagonal $\Delta : \mathcal{C} \to \mathcal{C}^2$. Then recall that in the polynomial $(x+y)^p$, every term except for $x^p$ and $y^p$ has coefficient a multiple of $p$. Therefore $(*\sqcup *)^p_{hC_p}$ can be identified with $BC_p \sqcup BC_p \sqcup \overline{S}$, where $\overline{S}$ is a discrete set constructed as follows: let $S$ be the subset of $\{x,y\}^p$ containing every point except for $x^p$ and $y^p$, and $\overline{S}$ is the quotient of $S$ under the free $C_p$ action on $S$ given by the obvious $C_p$ action on $\{x,y\}^p$.
Thus we have

$$\mathrm{Fun}((* \sqcup *) \wr C_p,\mathcal{C}) \simeq \mathcal{C}^{BC_p} \times \mathcal{C}^{BC_p} \times \prod_{\overline{\omega}\in\overline{S}} \mathcal{C}$$

Choosing a base point $e : * \to BC_p$, we have $q \wr C_p = (\mathrm{id}, \mathrm{id}, e, \dots, e)$, hence the right arrow of the diagram can be identified with $(\mathrm{id}, \mathrm{id}, e^*, \dots, e^*)$. Finally it suffices to identify the bottom arrow with a functor

$$\Phi : \mathcal{C} \times \mathcal{C} \to \mathcal{C}^{BC_p} \times \mathcal{C}^{BC_p} \times \prod \mathcal{C}$$

We write $\overline{\omega}(X,Y) = \omega_1(X,Y) \otimes \cdots \otimes \omega_p(X,Y)$, where $\omega$ is a preimage of $\overline{\omega}$ and $\omega_i(X,Y) = X$ if $\omega_i = x$, and $\omega_i(X,Y) = Y$ if $\omega_i = y$ (here $\omega_i$ is the $i$-th coordinate of $\omega$). Thus $\Phi$ is clearly given by

$$\Phi(X,Y) = (\Theta^p X, \Theta^p Y, \overline{\omega}(X,Y)_{\overline{\omega} \in \overline{S}})$$

**Corollary 21.3.9.** *Let $\mathcal{C}$ be a 0-semiadditively symmetric monoidal category. Then given $X, Y \in \mathcal{C}$*

$$\Theta^p(f+g) = \Theta^p(f) + \Theta^p(g) + \sum \int_e \overline{\omega}(f,g)$$

*Proof.* Directly by the above $\Theta^p$-square for $q$. □

Now we show the analogy of the above construction with some common features in arithmetics.

**Definition 21.3.10.** Let $R$ be a commutative ring. We define an additive $p$-derivation on $R$ (where $p$ is a prime number), is a map of sets $\delta : R \to R$ such that

$$\delta(x+y) = \delta(x) + \delta(y) + (x^p + y^p - (x+y)^p)/p$$

and $\delta(1) = 0$. We remark here that we should, as above, regard $(x^p + y^p - (x+y)^p)/p$ as an integral polynomial since all coefficients are naturally integers. We call the pair $(R, \delta)$ a semi-$\delta$-ring. A morphism of semi-$\delta$-rings are morphism of rings preserving the additive $p$-derivation.

*Remark* 33. We recall that a $p$-typical $\lambda$-ring is a ring $R$ equipped with an operation $\lambda : R \to R$ such that $\lambda/p$ is the Frobenius endomorphism of the $\mathbb{F}_p$-algebra $R/p$. Then we immediately see that the additive $p$-derivation above gives a $p$-typical $\lambda$-ring structure by letting $\lambda(x) = x^p + p\delta(x)$.

**Construction 21.3.11.** Let $\mathcal{C}$ be a 0-semiadditively symmetric monoidal category, then for $x \in \mathrm{coCAlg}(\mathcal{C})$ and $Y \in \mathrm{CAlg}(\mathcal{C})$, we have $\pi_0 \operatorname{Hom}(X,Y)$ a commutative rig (that is, a ring without additive inverses). We will then attempt to construct an additive $p$-derivation $\delta$ on this rig when $\mathcal{C}$ is 1-semiadditively symmetric monoidal through the following operation $\alpha$ which we will define later.

*Remark* 34. The terminology rig should come from rng, which is a ring without i, the unit.

We fix the notation by letting $e : * \to BC_p$ to be a choice of base point and $r : BC_p \to *$ the final map. Moreover, we let $\Theta^p$ denote the map $\Theta^p_* : \mathcal{C} \to \mathrm{Fun}(BC_p, \mathcal{C})$, the map sending an object $X$ to $X^{\otimes p}$ equipped with the permuting action of $C_p$ on the tensor.

**Construction 21.3.12.** Since $X$ is a cocommutative algebra in $\mathcal{C}$, the comultiplication map $X \to X^{\otimes p}$ factors through the homotopy fixed point $X \to (X^{\otimes p})^{hC_p} \simeq r_* \Theta^p(X)$. Similarly we have a map $r_! \Theta^p(Y) \simeq (Y^{\otimes p})_{hC_p} \to Y$. We denote by the dual of these maps $t_X : r^* X \to \Theta^p(X)$ and $m_Y : \Theta^p(Y) \to r^* Y$. By definition, since $re = \mathrm{id}_*$, we have $e^* t_X : X \to X^{\otimes p}$ the comultiplication and $e_* m_Y : Y^{\otimes p} \to Y$ the multiplication.
Then, for $g : \Theta^p(X) \to \Theta^p(Y)$, we denote $\overline{\alpha}(g)$ by the composition

$$X \xrightarrow{t_X^\vee} r_* \Theta^p(X) \xrightarrow{\mathrm{Nm}_r^{-1}} r_! \Theta^p(X) \xrightarrow{g} \xrightarrow{m_Y^\vee} Y$$

Then for $f : X \to Y$, we define $\alpha(f) = \overline{\alpha}(\Theta^p(f))$.

**Lemma 21.3.13.** *The map $\overline{\alpha} : \operatorname{Hom}(\Theta^p X, \Theta^p Y) \to \operatorname{Hom}(X,Y)$ constructed above is additive.*

*Proof.* We first notice that the map $r_*$ preserves products since it is taking the limit of a diagram. Thus the functor $r_*$ is additive, hence the map $\operatorname{Hom}(\Theta^p X, \Theta^p Y) \to \operatorname{Hom}(r_* \Theta^p X, r_* \Theta^p Y)$ is additive. Then $\overline{\alpha}$ is this additive map with a precomposition and a postcomposition, hence is additive. □

Moreover, $\alpha$ has naturality with functors and composition. Explicitly, we have

**Lemma 21.3.14.** *For $h : X' \to X$ a map in $\mathrm{coCAlg}(\mathcal{C})$ and $g : Y \to Y'$ a map in $\mathrm{CAlg}(\mathcal{C})$, for any $f : X \to Y$, we have $\alpha(g \circ f \circ h) \simeq g \circ \alpha(f) \circ h \in \mathrm{Hom}(X', Y')$.*
*For $F : \mathcal{C} \to \mathcal{D}$ a 1-semiadditive symmetric monoidal functor between 1-semiadditively symmetric monoidal categories, for any $X \in \mathrm{coCAlg}(\mathcal{C})$ and $Y \in \mathrm{CAlg}(\mathcal{C})$, we have $F : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(FX, FY)$ commutes with the operation $\alpha$.*

*Proof.* Direct. □

**Theorem 21.3.15.** *For any $X \in \mathrm{coCAlg}(\mathcal{C})$ and $Y \in \mathrm{CAlg}(\mathcal{C})$, we have $\alpha$ an additive p-derivation on $\pi_0 \mathrm{Hom}(X, Y)$. That is, for any $f, g : X \to Y$, we have*

$$\alpha(f + g) = \alpha(f) + \alpha(g) + ((f + g)^p - f^p - g^p)/p$$

*Proof.* Before proving the theorem, we first prove that for

$$h : e^*\Theta^p(X) \simeq X^{\otimes p} \to Y^{\otimes p} \simeq e^*\Theta^p(Y)$$

we have $\overline{\alpha}(\int_e h) : X \to Y$ the composition

$$X \xrightarrow{e^*t_X} X^{\otimes p} \xrightarrow{h} Y^{\otimes p} \xrightarrow{e^*m_Y} Y$$

Firstly, we see that $\int_e h$ is the composition

$$\Theta^p(X) \xrightarrow{u^e_*} e_*e^*\Theta^p(X) \xrightarrow{h} e_*e^*\Theta^p(Y) \xrightarrow{\mathrm{Nm}_e^{-1}} e_!e^*\Theta^p(Y) \xrightarrow{c^e_!} \Theta^p(Y)$$

Then after applying $\overline{\alpha}$, we get $\overline{\alpha} \int_e h$ the composition

$$\begin{tikzcd}
r_*\Theta^p(X) \arrow[r] & r_*e_*e^*\Theta^p(X) \arrow[r] & r_*e_*e^*\Theta^p(Y) \arrow[r] & r_*e_!e^*\Theta^p(Y) \arrow[r] & r_*\Theta^p(Y) \arrow[r] & r_!\Theta^p(Y) \arrow[d] \\
X \arrow[u] \arrow[r] & e^*\Theta^p(X) \arrow[u, equal] \arrow[r] & e^*\Theta^p(Y) \arrow[u, equal] \arrow[rrr] & & & Y
\end{tikzcd}$$

By the commutativity of this diagram we have proven the lemma. Then by corollary 21.3.9, we have

$$\alpha(f + g) = \alpha(f) + \alpha(g) + \overline{\alpha} \sum_{\varpi} \overline{\alpha}(\int_e \varpi(f, g))$$

By the lemma, we have $\overline{\alpha}(\int_e \varpi(f, g)) = f^i \otimes g^j$, where $i$ and $j$ are the times of $x$ and $y$ appearing in $\varpi$. □

In particular, we may apply the above to the following setting.

**Construction 21.3.16.** Let $\mathcal{C}$ be a symmetric monoidal category. Then we let

$$R_{\mathcal{C}} = \pi_0 \mathrm{Hom}(1_{\mathcal{C}}, 1_{\mathcal{C}})$$

Since $1_{\mathcal{C}}$ is trivially a commutative algebra and a cocommutative coalgebra, and the $\pi_0$ of the Hom space of a stable category is always an additive group, we have $R_{\mathcal{C}}$ a semi-$\delta$-ring if $\mathcal{C}$ is additionally stable and 1-semiadditive.

**Proposition 21.3.17.** *Let $\mathcal{C}$ be a 1-semiadditively symmetric monoidal category, then for any $f \in R_{\mathcal{C}}$, we have $\alpha(f) = \int_{BC_p} \Theta^p(f)$.*

*Proof.* We first notice that the action of $C_p$ on $\Theta^p(1_{\mathcal{C}}) \simeq 1_{\mathcal{C}}^{\otimes p}$ since $1_{\mathcal{C}}$ is the initial object in the category $\mathrm{CAlg}(\mathcal{C})$, and the action that permutes the factors in $1_{\mathcal{C}}^{\otimes p}$ is an action respecting the algebra structure of $1_{\mathcal{C}}^{\otimes p} \simeq 1_{\mathcal{C}}$. Consequently, we have the maps

$$t_{1_{\mathcal{C}}}^{\vee} : 1_{\mathcal{C}} \to r_* \Theta^p(1_{\mathcal{C}}) \simeq r_* r^* 1_{\mathcal{C}}$$

is the unit, and

$$m_{1_{\mathcal{C}}}^{\vee} : r_! r^* 1_{\mathcal{C}} \simeq r_! \Theta^p(1_{\mathcal{C}}) \to 1_{\mathcal{C}}$$

is the counit. Thus we have $\overline{\alpha}(g) = \int_{BC_p} g$ by unwinding the definitions, and hence the result. □

**Theorem 21.3.18.** *Let $\mathcal{C}$ be a $m$-semiadditively symmetric monoidal category for $m \geq 1$. Then for any $m$-finite space $A$, we have $\alpha(|A|) \simeq |A \wr C_p| \in R_{\mathcal{C}}$*

*Proof.* We define some map of spaces. We let $q : A \to *$ be the final map, $\pi : A \wr C_p \to BC_p$ be the map $q \wr C_p$, and $r : BC_p \to *$ the final map. Then we have

$$\alpha(|A|) \simeq \overline{\alpha}(\Theta^p(|A|)) \simeq \int_r \Theta^p(|A|) \simeq \int_r \Theta^p(\int_q \mathrm{id}_{1_{\mathcal{C}}}) \simeq \int_r \int_\pi \mathrm{id}_{1_{\mathcal{C}}} \simeq \int_{r\pi} \mathrm{id}_{1_{\mathcal{C}}} \simeq |A \wr C_p|$$

where the first three equality is given by definition and the fourth by theorem 21.3.7, and the fifth the naturality of integrals with respect to compositions. □

**Corollary 21.3.19.** *For any 1-semiadditively symmetric monoidal category and let $X \in \mathrm{coCAlg}(\mathcal{C})$ and $Y \in \mathrm{CAlg}(\mathcal{C})$, let $R = \pi_0 \mathrm{Hom}_{\mathcal{C}}(X, Y)$, we have $\alpha(1_R) \simeq |BC_p| \circ 1_R$.*

**Corollary 21.3.20.** *Firstly notice that the unit map $1_R$ is given by the composition $X \to 1_{\mathcal{C}} \to Y$, where the maps $\varepsilon : X \to 1_{\mathcal{C}}$ and $\eta : Y \to 1_{\mathcal{C}}$ are the counit and unit map given by the coalgebra and algebra structures on $X$ and $Y$. Then by the naturality of $\alpha$, we have*

$$\alpha(1_R) \simeq \alpha(\eta \circ 1_{R_{\mathcal{C}}} \circ \varepsilon) \simeq \eta \circ \alpha(1_{R_{\mathcal{C}}}) \circ \varepsilon$$

*Since obviously $1_{R_{\mathcal{C}}} \simeq |*| \in \mathrm{Hom}(1_{\mathcal{C}}, 1_{\mathcal{C}})$, we apply theorem 21.3.18 and we have*

$$\eta \circ \alpha(1_{R_{\mathcal{C}}}) \circ \varepsilon \simeq \eta \circ \alpha(|*|) \circ \varepsilon \simeq \eta \circ |BC_p| \circ \varepsilon \simeq |BC_p| \circ \eta \circ \varepsilon \simeq |BC_p| 1_R$$

*Here the only non-trivial equivalence is the second last equivalence, which is since $|BC_p|$ is a natural transformation.*

Now note that the only two obstructions preventing $\alpha$ from being an additive $p$-derivation is that the correction terms in the additivity has a sign error and $\alpha(1_R) \neq 0$. Therefore the following correction would produce an additive $p$-derivation.

**Construction 21.3.21.** Let $\mathcal{C}$ be a stable 1-semiadditively symmetric monoidal category with $X \in \mathrm{coCAlg}(\mathcal{C})$ and $Y \in \mathrm{CAlg}(\mathcal{C})$, we have $R = \pi_0 \mathrm{Hom}(X, Y)$ a ring, and we let $\delta : R \to R$ be given by $\delta(f) = |BC_p| f - \alpha(f)$. This is an additive $p$-derivation since $|BC_p| f$ is linear, and hence its additivity follows from theorem 21.3.15, and obviously $\delta(1_R) = 0$.

**Corollary 21.3.22.** *From lemma 21.3.14, we have for $F : \mathcal{C} \to \mathcal{D}$ a symmetric monoidal 1-semiadditive functor between stable 1-semiadditively symmetric monoidal categories, and given $X \in \mathrm{coCAlg}(\mathcal{C})$ and $Y \in \mathrm{CAlg}(\mathcal{C})$ we have*

$$F : \pi_0 \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \pi_0 \mathrm{Hom}_{\mathcal{D}}(FX, FY)$$

*a morphism of semi-$\delta$-rings.*

## 21.4 Detecting semiadditivity

In this section we finally present the core technique on detecting the semiadditivity of categories. We first give some basic properties of semi-$\delta$-rings.

**Definition 21.4.1.** For an element $x \in \mathbb{Q}$, we define $\delta(x) = (x - x^p)/p$. This is canonically (and uniquely) a additive $p$-derivation on any subring of $\mathbb{Q}$. For a ring $R$, We define $S_R$ to be the set of primes invertible in $R$, and we define $\mathbb{Q}_R$ to be $\mathbb{Q}[S_R^{-1}]$.

**Lemma 21.4.2.** *Let $(R, \delta)$ be a semi-$\delta$-ring, then for all $t \in \mathbb{Q}_R$ and $x \in R$ we have $\delta(tx) = t\delta(x) + \delta(t)x^p$.*

*Proof.* Consider the function $\varphi : \mathbb{Q}_R \to R$ given by $\varphi(t) = \delta(tx) - \delta(t)x^p$. Then by the additivity of $\delta$ and some calculation, we have $\varphi(t + s) = \varphi(t) + \varphi(s)$. Then since $\varphi(1) = \delta(x)$, we have $\varphi(t) = t\delta(x)$ since $\mathbb{Q}_R$ is a subring of $\mathbb{Q}$. □

**Lemma 21.4.3.** *Let $(R, \delta)$ be a $p$-local semi-$\delta$-ring, then any torsion element is nilpotent.*

*Proof.* Since $R$ is $p$-local, for any torsion element $x$ there is some $d$ such that $p^d x = 0$. Then by lemma 21.4.2, we have

$$0 = \delta(0) = \delta(p^d x) = p^d \delta(x) + \delta(p^d)x^p$$

Then multiplying $x$ on both sides we have $\delta(p^d)x^{p+1} = 0$. By elementary number theory we have $v_p(\delta(p^d)) = d-1$, hence we have $p^{d-1}x^{p+1} = 0$ since $R$ is $p$-local. Then repeat this process for $d$-times we have $x^{d(p+1)} = 0$. □

**Lemma 21.4.4.** *Let $(R, \delta)$ be a non-zero $p$-local semi-$\delta$-ring. Then the map $\varphi : \mathbb{Q}_R \to R$ is a monomorphism of semi-$\delta$-rings. In particular, the $\delta$ defined on $\mathbb{Q}_R$ is the unique additive $p$-derivation.*

*Proof.* Firstly we have $\varphi \circ \delta = \delta \circ \varphi$. Then if $\varphi$ is not injective, so is the initial morphism $\mathbb{Z} \to R$. Therefore $1 \in R$ is torsion. By lemma 21.4.3 we have 1 is nilpotent hence $R = 0$. □

**Corollary 21.4.5.** *For any non-zero $p$-local semi-$\delta$-ring $(R, \delta)$, either $p \in R^\times$ and $\mathbb{Q}_R = \mathbb{Q} \subseteq R$ or $p \notin R^\times$ and $\mathbb{Q}_R = \mathbb{Z}_{(p)} \subseteq R$, and $x \in \mathbb{Q}_R$ is invertible iff $v_p(x) = 0$.*

**Lemma 21.4.6.** *Let $(R, \delta)$ be a $p$-local semi-$\delta$-ring. Then the ideal $I_{\text{tor}} \subseteq R$ of torsion elements is closed under $\delta$.*

*Proof.* For $x \in I_{\text{tor}}$, we have some $p^d x = 0$. Thus $\delta(p^{d+1}x) = 0$, and by lemma 21.4.2, $\delta(p^{d+1}x) = p^{d+1}\delta(x) + \delta(p^{d+1})x^p$. Then since $\delta(p^{d+1})x^p = cp^d x^p = 0$, we have $p^{d+1}\delta(x) = 0$. Thus $\delta(x)$ is still torsion. □

**Proposition 21.4.7.** *We define $R^{\text{tf}}$ to be the ring $R/I_{\text{tor}}$. Then if $(R, \delta)$ is a $p$-local semi-$\delta$-ring, there is a unique additive $p$-derivation $\delta$ on $R^{\text{tf}}$, such that the quotient map $R \twoheadrightarrow R^{\text{tf}}$ is a morphism of semi-$\delta$-rings. Moreover, this quotient map detects invertibility (that is, the preimages of invertible elements are also invertible).*

*Proof.* We first note that for $x \in R$ and $y \in I_{\text{tor}}$, we have

$$\delta(x + y) - \delta(x) = \delta(y) + (x^p + y^p - (x + y)^p)/p$$

Then by lemma 21.4.6 we have $\delta(y) \in I_{\text{tor}}$ and $(x^p + y^p - (x + y)^p)/p = Cy \in I_{\text{tor}}$. Thus we have $\delta(x + I_{\text{tor}}) = \delta(x) + I_{\text{tor}}$ is a well-defined map of sets on $R^{\text{tf}}$. This is obviously the unique additive $p$-derivation on $R^{\text{tf}}$, and the quotient map detects invertibility since the kernel of the quotient map consists of torsion elements. □

**Corollary 21.4.8.** *Let $\mathcal{C}$ be a stable 1-semiadditively symmetric monoidal category with $X \in \mathrm{coCAlg}(\mathcal{C})$ and $Y \in \mathrm{CAlg}(\mathcal{C})$. Then let $R = \mathrm{Hom}(X, Y)$. Then every torsion element of $R$ is nilpotent, and if $\mathbb{Q} \otimes R = 0$ then $R = 0$.*

The main method we will use to detect semiadditivity is the method introduced by Hopkins-Lurie, which we will see below.

**Definition 21.4.9.** We define a 0-semiadditive category $\mathcal{C}$ to be $p$-local if for all $n$ coprime to $p$, we have $|[n]|_X$ is invertible.

**Theorem 21.4.10.** *A $(n-1)$-semiadditively presentably symmetric monoidal stable $p$-local category $\mathcal{C}$ is $n$-semiadditive iff $B^n C_p$ is $\mathcal{C}$-ambidextrous. In particular, if there is an $(n-1)$-finite $p$-space $A$ such that $\pi_{n-1}(A) \neq 0$ and $|A|_{1_\mathcal{C}}$ is an isomorphism, then $\mathcal{C}$ is $n$-semiadditive.*

*Proof.* $\Longrightarrow$ is obvious. For the converse, notice first by corollary 20.4.3, to exhibit $\mathcal{C}$ as $n$-semiadditive, it suffices to construct an $\mathbf{An}_n^n$-module on $\mathcal{C}$ compatible with the given $\mathbf{An}_{n-1}^{n-1}$-module structure. However, for an $n$-finite space $A$, from lemma 20.2.8, if we denote the projection $\pi : A \to \tau_{\leq n-1} A$, then we have $A = \varinjlim_{\tau_{\leq n-1} A} \pi^{-1}(x)$. In short, if the base space is $\mathcal{C}$-ambidextrous and so is all the fibers, then the total space is also $\mathcal{C}$-ambidextrous.
Therefore to define an $\mathbf{An}_n^n$-action on $\mathcal{C}$, it suffices to define the action for spaces with homotopy groups concentrated at $n$ since we may use these spaces to generate $\mathbf{An}_n^n$ by colimits indexed by $n-1$-finite spaces. Thus for an $n-1$-semiadditive category $\mathcal{C}$, to prove it to be $n$-semiadditive, it suffices to prove that all spaces with homotopy groups concentrated at the $n$-th degree is $\mathcal{C}$-ambidextrous.
Now, such spaces must be of the form $B^n G$ where $G$ is a group (Abelian if $n \geq 2$). For the case $n = 1$, we consider the sylow $p$-subgroup $H \subseteq G$. Then we have a fiber sequence $G/H \to BH \to BG$. Since $|G/H|$ is coprime to $p$, we have $G/H$ amenable, and hence by proposition 21.2.4, $BG$ is $\mathcal{C}$-ambidextrous if $BH$ is. Now since $H$ is a $p$-group, $H$ is a extension of finitely many copies of $C_p$. Thus since $BC_p$ is ambidextrous, so is $BH$. Hence $BG$ is $\mathcal{C}$-ambidextrous.
For $n \geq 2$, we have any Abelian group $G$ a direct sum of groups of the form $C_{q^n}$ (where $q$ is a prime), hence again it suffices to prove that all spaces of the form $B^n C_q$ are $\mathcal{C}$-ambidextrous. The case $q = p$ is provided by the assumption, and for the case $q \neq p$, we have by assumption $B^{n-1} C_q$ is $\mathcal{C}$-ambidextrous. Then since the diagonal map of $B^n C_q \to (B^n C_q)^2$ has fiber $B^{n-1} C_q$, the map $f : B^n C_q \to *$ is weakly ambidextrous. By the adjoint functor theorem, it suffices to prove the map $f_* : \mathrm{Fun}(B^n C_q, \mathcal{C}) \to \mathcal{C}$ preserves colimits. In fact we will actually show that $f_*$ is an equivalence of categories. That is, we show $\mathcal{C} \hookrightarrow \mathrm{Fun}(B^n C_q, \mathcal{C})$ is an equivalence. Note that this is equivalent to that for every space $K$ we have

$$\mathrm{Core}\,\mathrm{Fun}(K, \mathcal{C}) \simeq \mathrm{Core}\,\mathrm{Fun}(K, \mathrm{Fun}(B^n C_q, \mathcal{C}))$$

Hence replacing $\mathcal{C}$ by $\mathrm{Fun}(K, \mathcal{C})$, it suffices to prove

$$\mathrm{Core}\,\mathcal{C} \simeq \mathrm{Fun}(B^n C_q, \mathrm{Core}\,\mathcal{C})$$

Since $\mathrm{Core}\,\mathcal{C}$ is a space, we may prove the above for the $m$-truncations of $\mathrm{Core}\,\mathcal{C}$ and obtain the result by taking a colimit. The case $n = 1$ is trivial since $B^n C_q$ is 2-connective, hence every map from $B^m C_q$ to $\tau_{\leq 1} \mathrm{Core}\,\mathcal{C}$ is null. Assume we have

$$f_m^* : \tau_{\leq n} \mathrm{Core}\,\mathcal{C} \simeq \mathrm{Fun}(B^m C_q, \tau_{\leq n} \mathrm{Core}\,\mathcal{C})$$

Then to prove $f_{m+1}^*$ is an equivalence, we consider the diagram

$$\begin{tikzcd}
\tau_{\leq m+1} \mathrm{Core}\,\mathcal{C} \arrow[r] \arrow[d] & \mathrm{Fun}(B^m C_q, \tau_{\leq m+1} \mathrm{Core}\,\mathcal{C}) \arrow[d] \\
\tau_{\leq m} \mathrm{Core}\,\mathcal{C} \arrow[r] & \mathrm{Fun}(B^m C_q, \tau_{\leq m} \mathrm{Core}\,\mathcal{C})
\end{tikzcd}$$

It suffices to prove $f_{m+1}^*$ is an equivalence on all fibers of $C \in \tau_{\leq m}\operatorname{Core}\mathcal{C}$. Notice that the fiber of the projection $\tau_{\leq m+1}\operatorname{Core}\mathcal{C} \to \tau_{\leq m}\operatorname{Core}\mathcal{C}$ at $C$ is the Eilenberg-Maclane space $K(\pi_m \operatorname{Aut}_{\mathcal{C}}(C), m+1)$. Since the information of automorphisms is contained in the information of endomorphisms, it suffices to prove

$$K(\operatorname{Ext}_{\mathcal{C}}^{-m}(C,C), m+1) \to \operatorname{Fun}(B^m C_p, K(\operatorname{Ext}_{\mathcal{C}}^{-m}(C,C), m+1))$$

is an equivalence. This is the vanishing of the $m+1$-th reduced cohomology $\tilde{H}^{m+1}(B^m C_p; \operatorname{Ext}_{\mathcal{C}}^{-m}(C,C))$ which is true since $\operatorname{Ext}_{\mathcal{C}}^{-m}(C,C)$ is $p$-local.
For the final assertion, notice that since $A$ is a $(n-1)$-finite $p$-space, so we have a fiber sequence $A \to B \to B^n C_p$ where $b$ is also $(n-1)$-finite. The rest follows from proposition 21.2.4 (note that $A$ is amenable since $|A|_{1_{\mathcal{C}}}$ is an isomorphism). □

**Corollary 21.4.11.** *Let $\mathcal{C}$ be an $(n-1)$-semiadditively presentably symmetric monoidal stable $p$-local category. We define $A$ to be an $h$-good space if it is a connected $m$-finite $p$-space with $\pi_m(A) \neq 0$ and that $h(|A|)$ is rational. Then let $h : R_{\mathcal{C}} \to S$ (recall $R_{\mathcal{C}}$ is the semi-$\delta$-ring $\pi_0 \operatorname{Hom}(1_{\mathcal{C}}, 1_{\mathcal{C}})$ equipped with the $\delta$-operation defined by $\alpha$) be a morphism of semi-$\delta$-rings detecting invertibility and such that $h(|BC_p|)$ and $h(|A|)$ are rational and non-zero for an $h$-good space $A$, then $\mathcal{C}$ is $n$-semiadditive.*

*Proof.* Then by theorem 21.4.10, it suffices to find a $h$-good space $X$ such that $|X|$ is invertible. Firstly by assumption we have $h(|A|)$ is $h$-good. If $p \in S^\times$ we have $B^{n-1}C_p$ satisfying our desired conditions and we are done. So we assume $p \notin S^\times$. In this case, we show that there is an $h$-good space $A$ with $v_p(h(|X|)) = 0$.
We have $0 \leq v_p(h(|A|)) < \infty$, so it suffices to find for any $h$-good space $A$ another $h$-good space $A'$ such that $v_p(h(|A|)) - 1 = v_p(h(|A'|))$. We prove $A' = A \wr C_p$ satisfies the two conditions. Firstly, we have

$$h(|A \wr C_p|) = h(|BC_p \times A|) - h(\delta|A|) = h(|BC_p|)h(|A|) - h(\delta|A|)$$

Then since $h(|BC_p|)$, $h(|A|)$ and $h(\delta|A|)$ are all rational, we have $h(|A \wr C_p|)$ rational. Moreover, obviously $A \wr C_p$ is $m$-finite $p$-space with $\pi_m(A) \neq 0$, hence $A \wr C_p$ is $h$-good. Now, since $h(|BC_p|)$ is rational and $p \notin S^\times$, we have $v_p(h(|A|)) \leq v_p(h(|BC_p \times A|))$. Finally by elementary number theory we have $v_p(h(\delta|A|)) = v_p(h(|A|)) - 1$, so we have prove the result, and the assertion follows by induction. □

We then present the method that we will use later to apply the theory of semiadditivity to chromatic localized categories.

**Theorem 21.4.12.** *Let $1 \leq m \leq \infty$ and $F : \mathcal{C} to \mathcal{D}$ be a colimit preserving symmetric monoidal functor between presentably symmetric monoidal stable $p$-local categories. Assume that $\mathcal{C}$ is 1-semiadditive, and the map $\varphi_F : R_{\mathcal{C}} \to R_{\mathcal{D}}$ detects invertibility, and finally for $0 \leq k < m$, if $B^k C_p$ is dualizable in $\mathcal{D}$ (recall the notion of dualizability of spaces in definition 21.2.19) then the image of $\dim_{\mathcal{D}}(B^k C_p)$ in $R_{\mathcal{D}}^{\mathrm{tf}}$ is rational and non-zero, then both $\mathcal{C}$ and $\mathcal{D}$ are $m$-semiadditive.*

*Proof.* By corollary 21.2.15, it suffices to show that $\mathcal{C}$ is $m$-semiadditive. We prove by induction on $k$, where the condition is that for $0 \leq i < k$ we have $|B^i C_p|_{\mathcal{D}}$ in $R_{\mathcal{D}}^{\mathrm{tf}}$ are rational and non-zero, and that $\mathcal{C}$ is $k$-semiadditive. The case where $k = 1$ is by assumption. For $k \geq 2$, by corollary 21.2.21, we have $B^k C_p$ dualizable and

$$\dim_{\mathcal{D}}(B^k C_p) = |B^k C_p|_{\mathcal{D}} |B^{k-1} C_p|_{\mathcal{D}} \in R_{\mathcal{D}}^{\mathrm{tf}}$$

Therefore by the assumptions we have $\dim_{\mathcal{D}}(B^k C_p)$ is rational and non-zero. Therefore by the inductive hypothesis, the image of $|B^{k-1} C_p|_{\mathcal{D}}$ in $R_{\mathcal{D}}^{\mathrm{tf}}$ is rational and non-zero hence so is $|B^k C_p|_{\mathcal{D}}$. Then since the map $R_{\mathcal{D}} \to R_{\mathcal{D}}^{\mathrm{tf}}$ detects invertiblity since $\mathcal{D}$ is $p$-local, $|B^k C_p|_{\mathcal{C}}$ is rational and non-zero. □

To better apply the above theorem, we present a setting where the map $\varphi_F$ induced by $F : \mathcal{C} \to \mathcal{D}$ detects invertibility.

**Definition 21.4.13.** Let $F : \mathcal{C} \to \mathcal{D}$ be a colimit preserving monoidal functor between stably presentably monoidal categories nil-conservative if for every ring $R \in \mathrm{Alg}(\mathcal{C})$ we have $F(R) \simeq 0$ implies $R = 0$.

Directly by definition we have

**Lemma 21.4.14.** *1. Any conservative functor is nil-conservative.*

*2. The composition of nil-conservative functors is nil-conservative.*

*3. For $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, if $GF$ is nil-conservative then so is $F$.*

**Lemma 21.4.15.** *Let $\mathcal{C}$ be a stably presentably monoidal category, and let $R, S \in \mathrm{Alg}(\mathcal{C})$, then for every cofiber sequence $X \to Y \to Z \in {}_S\mathbf{Mod}_R(\mathcal{C})$, if two out of $X, Y, Z$ are right dualizable then so is the third.*

*Proof.* Suppose $X$ and $Y$ are right dualizable. We let $\mathcal{M}$ be a category with a left $\mathcal{C}$-action on it, that is, $\mathcal{M} \in \mathbf{LMod}_{\mathcal{C}}(\mathbf{Pr}^{\mathrm{L}})$. Thus we have a functor

$$X \otimes_R (-) : \mathbf{LMod}_R(\mathcal{M}) \to \mathbf{LMod}_S(\mathcal{M})$$

This construction is obviously functorial with respect to functors for $\mathcal{M}$. Notice that $X$ is right dualizable iff for every $\mathcal{M}$, the functor $X \otimes_R (-)$ has a left adjoint, and for every functor $U : \mathcal{M}' \to \mathcal{M}''$, the Beck-Chevalley map $F''_X U \to U F'_X$ obtained from the adjunction and naturality is an isomorphism. Then for every $\mathcal{M}$ we have a cofiber sequence of functors

$$X \otimes_R - \to Y \otimes_R - \to Z \otimes_R -$$

This induces a cofiber sequence of adjoint functors $F_Z \to F_Y \to F_X$. Thus we have a diagram of Beck-Chevalley maps

$$\begin{tikzcd} F''_Z U \arrow[r] \arrow[d] & F''_Y U \arrow[r] \arrow[d] & F''_X U \arrow[d] \\ U F'_Z \arrow[r] & U F'_Y \arrow[r] & U F'_X \end{tikzcd}$$

Thus if the second and third vertical map are equivalences, so is the first. □

**Lemma 21.4.16.** *A colimit preserving monoidal functor $F : \mathcal{C} \to \mathcal{D}$ of stably presentably monoidal categories is nil-conservative iff for every $S \in \mathrm{Alg}(\mathcal{C})$, the induced functor $F : \mathbf{LMod}_S(\mathcal{C}) \to \mathbf{LMod}_{F(S)}(\mathcal{D})$ is conservative when restricted to the subcategory of right dualizable modules.*

*Proof.* $\Longleftarrow$ is obvious since every ring is right dualizable over itself as a left module.
$\Longrightarrow$ : let $f : N_1 \to N_2$ be a map of right dualizable left $S$-modules and let $M$ be the cofiber of $f$, and by lemma 21.4.15 $M$ is also right dualizable. Thus we show that if $FM = 0$ then $M = 0$. Let $M^\vee$ denote the right dual of $M$, we have $M^\vee \otimes_S M \simeq \mathrm{Map}_S(M, M)$. Thus $F(\mathrm{Map}_S(M, M)) \simeq F(M^\vee \otimes_S M) \simeq 0$ since $F$ is monoidal. Thus $\mathrm{Hom}_S(M, M) \simeq 0$, and hence $M \simeq 0$. □

**Corollary 21.4.17.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a nil-conservative $m$-semiadditively symmetric monoidal functor between $m$-semiadditively symmetric monoidal stable presentable $p$-local categories. Then the map $\varphi_F : R_{\mathcal{C}} \to R_{\mathcal{D}}$ detects invertibility. In particular if $A$ is $\mathcal{C}$-ambidextrous and $\mathcal{D}$-amenable the $A$ is $\mathcal{C}$-amenable.*

*Proof.* This follows from lemma 21.4.16 since $1_{\mathcal{C}}$ is dualizable. □

## 21.5 Applications in the chromatic settings

# Chapter 22

# Semiadditive Height